GI12/4C59 – Homework #1 (Due 12am, October 25, 2005)

Aim: to get familiarity with basic probability and practicing mathematical reasoning. Presentation, clarity, and synthesis of exposition will be taken into account in the assessment of these exercises.

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1. **[30 pts]** A box contains w white marbles and b black marbles. Two marbles are taken from the box, without replacement. Prove that the probability that the first ball is white is equal to the probability that the second ball is white.
2. **[20 pts]** You meet Fred. Fred tells you he has two brothers, Alf and Bob. What is the probability that Fred is older than Bob?
3. **[20 pts]** Fred tells you that he is older than Alf. Now, what is the probability that Fred is older than Bob? (That is, what is the conditional probability that $F > B$ given that $F > A$)?
4. **[30 pts]** We discussed the fabled "Monty Hall Problem" in class. Now you are going to describe the conclusion (be sure and make a clear, thorough discussion of your result):

   "Let's Make A Deal" host Monty Hall gives a game show contestant the choice between three doors. Behind two of the doors are junk, and behind the third is a prize.

   After the contestant selects a door, Monty reveals junk behind one of the remaining doors.

   Then, he gives the contestant a chance to keep the door that was initially selected, or to switch to the remaining, unopened door.

   Should the contestant switch, stay with the initial selection, or does it not matter?

Model Solutions:

1. **[30 pts]** A box contains *w* white marbles and *b* black marbles. Two marbles are taken from the box, without replacement. Prove that the probability that the first ball is white is equal to the probability that the second ball is white.

Solution:
The probability of the first marble selected being white is simply *w/(w+b)*. When the second marble is selected, there are two possibilities. Case A is when the first marble was white. This leaves *w-1* white marbles, and *b* black marbles. So the probability that the second marble is white is *(w-1)/(w+b-1)*. Case B is when the first marble was black. This leaves *w* white marbles, and *b-1* black marbles. So the probability that the second marble is white is *w/(w+b-1)*. Therefore, the total probability that the second marble is white is:

$$P = \left(\frac{w}{w+b}\right)\left(\frac{w-1}{w+b-1}\right) + \left(\frac{b}{w+b}\right)\left(\frac{w}{w+b-1}\right) =$$

$$\frac{w^2 - w + bw}{(w+b)(w+b-1)} = \frac{w(w+b-1)}{(w+b)(w+b-1)} = \frac{w}{w+b}$$

Which is the same as for the first marble.

2. **[20 pts]** You meet Fred. Fred tells you he has two brothers, Alf and Bob. What is the probability that Fred is older than Bob?

There are 3!=6 possible birth orders:
ABF, AFB, BAF, BFA, FAB, FBA
Which are equally likely. F precedes A in three of these, so the probability (as is intuitive) is ½.

3. **[20 pts]** Fred tells you that he is older than Alf. Now, what is the probability that Fred is older than Bob? (That is, what is the conditional probability that *F* > *B* given that *F* > *A*)?

The extra knowledge restricts us to three cases:
BFA, FAB, FBA
Which are, once again, equally likely given the information we have. F precedes B in two of these cases, so the probability is 2/3.

We can confirm this analysis with Bayes rule (without the circular reasoning that you may have encountered in trying to work this out) by properly characterizing the knowledge in each event state that occurs. Let's call *F=O* the probability that *F* is the oldest (*F>A & F>B*). We want to know:

$$P(F = O \mid F > A) = \frac{P(F > A \mid F = O)P(F = O)}{P(F > A)}$$

$$= \frac{(1)(1/3)}{(1/2)} = \frac{2}{3}$$

4. **[30 pts]** We discussed the fabled "Monty Hall Problem" in class. Now you are

going to describe the conclusion (be sure and make a clear, thorough discussion of your result):

> "Let's Make A Deal" host Monty Hall gives a game show contestant the choice between three doors. Behind two of the doors are junk, and behind the third is a prize.
> After the contestant selects a door, Monty reveals junk behind one of the remaining doors.
> Then, he gives the contestant a chance to keep the door that was initially selected, or to switch to the remaining, unopened door.
> Should the contestant switch, stay with the initial selection, or does it not matter?

There are basically two (equally valid, but different) approaches to describing the result in words. The first is the Bayesian approach outlined below. It rates belief in the past condition that the prize is behind the remaining door, versus the condition that the prize is behind the originally selected door, given the evidence (of the door Monty selected).

Let's call the door first picked by the contestant A. That leaves doors *B* and *C* that Monty may open. Remember: he will only open a door that does not have the prize behind it. Let's say that if Monty opens door *C* the event $M{=}C$ has occurred. Let's say that if the prize is behind door *A* the event $R{=}A$ has occurred.
Let's say Monty opens door *B*. The contestant should switch doors if $P(R{=}C|M{=}B) > P(R{=}A|M{=}B)$. Note that $P(R{=}C|M{=}B) + P(R{=}A|M{=}B) = 1$. Therefore, the contestant should switch if $P(R{=}C|M{=}B) > (1/2)$, and it doesn't matter if $P(R{=}C|M{=}B) = (1/2)$

$$P\left(R = C \mid M = B\right) = \frac{P\left(M = B \mid R = C\right) P\left(R = C\right)}{P\left(M = B\right)}$$

The first term is the probability that Monty opens door *B* if the prize is behind door *C*. *This probability is 1*, because Monty can only open a door that does not have a prize behind it, and he can't open the door that the contestant has selected already.
The second term is the unconditional probability that the reward is behind door *C*. This is 1/3. The denominator is the probability that Monty opens door *B* unconditionally. Since we have already declared that door *A* is selected by the contestant, this probability is ½. Therefore:

$$P\left(R = C \mid M = B\right) = \frac{P\left(M = B \mid R = C\right) P\left(R = C\right)}{P\left(M = B\right)}$$

$$= \frac{(1)(1/3)}{(1/2)} = \frac{2}{3}$$

The other possible description involves the mean value interpretation of probability: If this experiment was conducted over and over again, would switching yield more "wins" on average, or not? In this interpretation, one third of the time the contestant will select the correct door on the first try ($R{=}A$). If this is the case, switching is sure to be wrong: not switching always yields a win. But *two thirds of the time the contestant first chooses the wrong door*. And in these cases, switching the door always yields a win. So, two thirds of the time switching will give a win, on average.