1 Introduction

This coursework asks you to consider recent theoretical ideas about solution of undersampled (i.e. non-unique) inverse problems with constraints on the *sparsity* of the solutions. You are asked to read the suggested paper and provide critical comment upon it, and to test some of the claims with numerical experiments.

2 Reconstruction with Sparsity Constraints

2.1 Critical Analysis

Consider the paper

[1] E. J. Candès, J. Romberg and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. Comm. Pure Appl. Math., **59** 1207-1223. (2005)
which you can find at

`https://statweb.stanford.edu/~candes/publications/downloads/StableRecovery.pdf`

How many citations does paper [1] have, and how influential do you think it has been ? Find an example of a later paper citing [1], and comment on the later paper's impact.

Read [1] and supporting references if needed and use it to provide an answer to this question:

"How and under what circumstances can a signal f be exactly reconstructed from a discrete set of samples"?

2.2 Numerical Experiments

You are asked to reproduce the results in part 3 of this paper (first experiment : figure 2(a)-(b))[1]). This solves a problem

$$\min \|\mathbf{f}\|_1, \text{ subject to } \|\mathsf{A}\mathbf{f} - \mathbf{y}\|_2 \leq \epsilon, \tag{1}$$

where $\mathbf{y}$ is a 300×1 vector A is a 300×1024 matrix and $\mathbf{f}$ is a 1024×1 vector. What is the singular spectrum of the matrix A as described in the paper? Compute a solution to (1) with and without noise in your data. The problem without noise is known as *Basis Pursuit*. The problem with noise is known as as *Basis Pursuit De-Noising*.

Construct another matrix with the same singular vectors as A but where the singular spectrum is changed to $\exp(-k/100)$. Repeat the experiments with this new matrix. Is the recovery of $\mathbf{f}$ still stable?

[1] In the journal version of the paper, this is figure 3.1(a)-(b)

3 Hints

In each case, you first have to construct a matrix A. One way is to construct an arbitrarily orientated orthonormal basis U of size 300×300, another arbitrarily orientated orthonormal basis V, and a designed singular value matrix W, after which you can build $\mathsf{A} = \mathsf{U}\mathsf{W}\mathsf{V}^{\mathrm{T}}$. Note that the size of W and V depends on the particular implementation; either $\mathsf{W} = 300 \times 300$ and $\mathsf{V} = 1024 \times 300$, or $\mathsf{W} = 300 \times 1024$ and $\mathsf{V} = 1024 \times 1024$ are correct.

After constructiong A we want to solve the problem in equation (1). In principle this is the same as you did in the first exercise in Coursework 1, but with 300 equations and 1024 unknowns instead of one equation and two unknowns. So you could use the same methods as you did there. However, it is likely that those methods will not work well and will take a very long time to find a solution. Therefore we recommend you use one of the more specialised methods as follows :

Matlab: You can create two random orthogonal bases as follows:

```
A = rand(300,1024);
U = orth(A);
V = orth(A');
```

For the optimisation, use the "L1 magic" code located at:

`https://statweb.stanford.edu/~candes/software/l1magic/`

To solve the above optimisation problem (1), you can use either the function `l1eq_pd` or `l1qc_logbarrier`

Python: You can create two random orthogonal bases as follows:

```
A = numpy.random.randn(300,1024);
U = scipy.linalg.orthogonal(A);
V = scipy.linalg.orthogonal(numpy.transpose(A));
```

You can use the package `cvxpy` to solve the optimisation problem. Follow the syntax as described at

`https://www.cvxpy.org/`

with the objective and constraint as in equation (1).

4 Report

You should write a short report (probably no more that 4 pages, including figures). The report should answer all the questions asked in section 2 and include your results from section 2.2.

If desired, you can (optionally) include code you used printed as an appendix to your report, but it will not count towards the 4 pages.