# Justification logic for constructive modal logic

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# The big picture

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**Justification logic:**
Gödel:
What is the classical provability semantics of intuitionistic logic?
Artemov:
*Logic of Proofs* gives an operational view of this S4 type of provability.

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Extensions: realisation of logics below and above S4

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**Intuitionistic variants:** Some investigations toward

- realisation theorems (Artemov/Steren and Bonelli),
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but where the modal language is restricted to the $\Box$ modality.

However, intuitionistically $\Diamond$ cannot simply be viewed as the dual of $\Box$.

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**Intuitionistic modal logic?**
The program: represent the operational side of the intuitionistic $\Diamond$.

The focus: on constructive versions of modal logic.

# Constructive modal logic

**Formulas:** $A ::= \bot \mid a \mid A \wedge A \mid A \vee A \mid A \supset A$

**Logic** CK**:** Intuitionistic Propositional Logic

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$$+ \quad \begin{array}{l} \mathsf{k}_1: \ \Box(A \supset B) \supset (\Box A \supset \Box B) \\ \mathsf{k}_2: \ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \end{array} \qquad + \quad \text{necessitation:} \quad \frac{A}{\Box A}$$

(Wijesekera/Bierman and de Paiva/Mendler and Scheele)

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Justification logic adds proof terms directly inside its language.

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In the constructive version, we also add witness terms into the language.

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# Justification logic

**Modal formulas:** $A ::= \bot \mid a \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A$

**Justification formulas:** $A ::= \bot \mid a \mid A \wedge A \mid A \vee A \mid A \supset A \mid t : A$

**Grammar of terms:**

$$t ::= \quad c \quad \mid \quad x \quad \mid \quad (t \cdot t) \quad \mid \quad (t + t) \quad \mid \quad !\,t$$

$c$ : proof constants
$x$ : proof variables
$\cdot$ : application
$+$ : sum
$!$ : *proof checker*

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$\alpha$ : witness variables

$\star$ : execution

$\sqcup$ : disjoint witness union

# Justification logic for constructive modal logic

**Axiomatisation** JCK**:**

taut: Complete finite set of axioms for intuitionistic propositional logic

$\mathsf{jk}_\Box$: $t : (A \supset B) \supset (s : A \supset t \cdot s : B)$

sum: $s : A \supset (s + t) : A$ and $t : A \supset (s + t) : A$

$$\mathsf{mp}\ \frac{A \supset B \quad A}{B} \qquad\qquad \mathsf{ian}\ \frac{A \text{ is an axiom instance}}{c_1 : \ldots c_n : A}$$

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# The machinery

**Application:** $\mathsf{jk}_{\Box} : t:(A \supset B) \supset (s:A \supset t \cdot s:B)$

If $t$ is a proof of $A \supset B$ and $s$ is a proof of $A$, then $t \cdot s$ is a proof of $B$.

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**Witness execution:** $\mathsf{jk}_{\Diamond} : t:(A \supset B) \supset (\mu:A \supset t \star \mu:B)$

If $t$ is a proof of $A \supset B$ and $\mu$ is a witness for $A$, then the same model denoted $t \star \mu$ is also a witness for $B$.

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**Sum and union:** $s:A \supset (s+t):A$, $\mu:A \supset (\mu \sqcup \nu):B, \ldots$
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**Iterated axiom necessitation and modus ponens:**

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**Lifting Lemma:**

- If $A_1, \ldots, A_n \vdash_{\mathsf{JCK}} B$, then there exists a proof term $t(x_1, \ldots, x_n)$ such that, for all terms $s_1, \ldots, s_n$

$$\vdash_{\mathsf{JCK}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \supset t(s_1, \ldots, s_n) : B$$

- If $A_1, \ldots, A_n, C \vdash_{\mathsf{JCK}} B$, then there exists a witness term $\mu(x_1, \ldots, x_n, \beta)$ such that, for all terms $s_1, \ldots, s_n$ and $\nu$

$$\vdash_{\mathsf{JCK}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \wedge \nu : C \supset \mu(s_1, \ldots, s_n, \nu) : B$$

# Correspondence

**Forgetful projection:** If $\vdash_{\mathsf{JCK}} F$, then $\vdash_{\mathsf{CK}} F^\circ$,

where $(\cdot)^\circ$ maps justification formulas onto modal formulas, in particular:

$$(t : A)^\circ := \Box A^\circ \qquad\qquad (\mu : A)^\circ := \Diamond A^\circ$$

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Standard solution:
Consider a proof of the modal theorem in a cut-free sequent calculus.

# Sequent calculus for modal logic

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**Sequent system** LCK**:**

$$\text{id}\ \frac{}{\Gamma, a \Rightarrow a} \qquad \bot_{\mathsf{L}}\ \frac{}{\Gamma, \bot \Rightarrow C}$$

$$\vee_{\mathsf{L}}\ \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\wedge_{\mathsf{L}}\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\supset_{\mathsf{L}}\ \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \qquad \supset_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

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$$\mathsf{k}_{\Box}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Box A} \qquad \mathsf{k}_{\Diamond}\ \frac{\Gamma, B \Rightarrow A}{\Box\Gamma, \Delta, \Diamond B \Rightarrow \Diamond A}$$

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$$k_\Box\ \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Box A} \qquad k_\Diamond\ \frac{\Gamma, B \Rightarrow A}{\Box\Gamma, \Delta, \Diamond B \Rightarrow \Diamond A}$$

**Soundness and completeness:** $\vdash_{\mathsf{CK}} A$ iff $\vdash_{\mathsf{LCK}} \Rightarrow A$.

# Main theorem

**Realisation:** If $\vdash_{\mathsf{LCK}} A'_1, \ldots, A'_n \Rightarrow C'$, a modal sequent, then there is a normal realisation $A_1, \ldots A_n \Rightarrow C$ of $A'_1, \ldots, A'_n \Rightarrow C'$ such that $\vdash_{\mathsf{JCK}} (A_1 \wedge \ldots \wedge A_n) \supset C$.

- if $t : A/\mu : A$ is a negative subformula of $A_1, \ldots A_n \Rightarrow C$, then $t/\mu$ is a proof/witness variable, and all these variables are pairwise distinct.

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- if $t : A/\mu : A$ is a negative subformula of $A_1, \ldots A_n \Rightarrow C$, then $t/\mu$ is a proof/witness variable, and all these variables are pairwise distinct.

The proof goes along the lines of that for the $\Box$-only fragment.

The operation $\sqcup$ on witness terms plays the same role as the operation $+$ on proof terms, i.e. to handle contractions of modal formulas.

# Extensions

$$
\begin{tikzcd}
\mathsf{CT} \arrow[r, dash] \arrow[d, dash] & \mathsf{CS4} \arrow[d, dash] & \\
\mathsf{CD} \arrow[r, dash] \arrow[d, dash] & \mathsf{CD4} \arrow[r, dash] \arrow[d, dash] & \mathsf{CD45} \arrow[d, dash] \\
\mathsf{CK} \arrow[r, dash] & \mathsf{CK4} \arrow[r, dash] & \mathsf{CK45}
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\begin{array}{rl}
\mathsf{d}: & \Box A \supset \Diamond A \\
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In particular, the $\Box$-version of 4 requires the proof checker operator !

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In particular, the $\Box$-version of 4 requires the proof checker operator !

$$\mathsf{j4}_{\Box} : t : A \supset\, !t : t : A$$

but *a priori* no additional operation for the $\Diamond$-version of 4.

$$\mathsf{j4}_{\Diamond} : \mu : \nu : A \supset \nu : A$$

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5: & (\Diamond A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset \Box A)
\end{array}$$

No other operation on witness terms outside execution and disjoint union.
In particular, the $\Box$-version of 4 requires the proof checker operator !

$$\mathsf{j4}_{\Box} : t : A \supset \, !\, t : t : A$$

but *a priori* no additional operation for the $\Diamond$-version of 4.

$$\mathsf{j4}_{\Diamond} : \mu : \nu : A \supset \nu : A$$

We think that the method here could be further extended,
but we would need to prove cut-elimination for the other systems.

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**Thank you. Let's discuss!**