Mathematics for M.Sc. CS and FC
Boolean Algebra Exercises
With Answers

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1. Let $S = \{x, y\}$ be a set with two elements. The *power set* $\wp(S)$ of S consists of all subsets of S, namely

   $$\emptyset, \{x\}, \{y\}, \{x, y\}$$

   These four elements form a boolean algebra. We interpret negation $-$ as the complement of a set (in S), so $-\{x\} = \{y\}$, for example. Conjunction, $\cdot$, is interpreted as intersection of sets, so $\{x\} \cdot \{y\} = \emptyset$. And disjunction, $+$, is interpreted as union so $\{x\} + \{y\} = \{x, y\}$, for example.

   (a) Write down the definitions of $+, -$ for all elements.
   Answer: Write 0 *for* $\emptyset$ *and write* 1 *for* $\{x, y\}$.

$+$	0	$\{x\}$	$\{y\}$	1
0	0	$\{x\}$	$\{y\}$	1
$\{x\}$	$\{x\}$	$\{x\}$	1	1
$\{y\}$	$\{y\}$	1	$\{y\}$	1
1	1	1	1	1

   and

   $$- : \begin{cases} 0 & \to & 1 \\ \{x\} & \to & \{y\} \\ \{y\} & \to & \{x\} \\ 1 & \to & 0 \end{cases}$$

   so $+$ *is union of sets and* $-$ *is complement of a set.*

   (b) Define a boolean set algebra, as above, where S is a three element set $\{x, y, z\}$. How many elements are there in the boolean set algebra?
   Answer:8

   (c) How many elements are there in a boolean set algebra based on a set S with n elements?
   Answer:2^n

2. Can define other operators from $+, -$.

   $$\begin{aligned} x\ NOR\ y &= -(x + y) \\ x\ NAND\ y &= -(x \cdot y) \end{aligned}$$

   (a) Write down the operator tables for NOR and NAND.
   Answer:

NOR	0	1
0	1	0
1	0	0

$NAND$	0	1
0	1	1
1	1	0

   (b) Write down an expression equivalent to $-x$ only using the NAND operator.
   Answer:$xNANDx$

   (c) Write down an expression equivalent to $x + y$ only using the NAND operator.
   Answer:

   $$\begin{aligned} x + y &= -(\bar{x}.\bar{y}) \\ &= \bar{x} NAND \bar{y} \\ &= (xNANDx)NAND(yNANDy) \end{aligned}$$

   (d) Express $-$ and $+$ only using the NOR operator.
   Answer:$-x = xNORx$.

   $$\begin{aligned} x + y &= -(xNORy) \\ &= (xNORy)NOR(xNORy) \end{aligned}$$

3. Find a sum of products equivalent to the following boolean expressions.

(a) $a \cdot (b + c)$
Answer:$a.b + a.c$

(b) $(a + b) \cdot (\bar{a} + \bar{b})$
Answer:$a.\bar{a} + a.\bar{b} + b.\bar{a} + b.\bar{b} = a.\bar{b} + b.\bar{a}$

(c) $-(a + b \cdot c)$.
Answer:$\bar{a}.\bar{b} + \bar{c}$

4. For each of the following boolean expressions, write down a truth table. Use this to determine whether the expression is *valid* (true under all valuations) or not. Also, use your truth table to find a sum of products equivalent to the given expression.

(a) $p \cdot (p + -q)$
Answer:$p + p.\bar{q}$, *not valid, e.g. if* p *is false, the expression is false*

(b) $-(p \cdot -q) \rightarrow ((p \cdot q) + r)$
Answer:$p.q + r + p.\bar{q}$ *not valid*

(c) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \cdot q) + r)$
Answer:$p.q + r + p.q.\bar{r}$ *not valid*

(d) $p \rightarrow (q \rightarrow p)$.
Answer:$p + \bar{q} + \bar{p}$. *Valid (true, regardless of truth values for* p, q*)*

5. Using only the element 0 and the operator $\rightarrow$, find an expression equivalent to each of the following.

(a) $-x$
Answer: $x \rightarrow 0$

(b) $x + y$
Answer:$(x \rightarrow 0) \rightarrow y$

(c) $x \cdot y$
Answer:$((x \rightarrow 0) \rightarrow y) \rightarrow 0$

(d) $(x \cdot y) + (\bar{x} \cdot \bar{y})$.
Answer:$((x \rightarrow 0) \rightarrow y) \rightarrow (((x \rightarrow 0) \rightarrow y) \rightarrow 0)$

6. Write down the statement that is *dual* to the following. "If $x+y = 0$ then $x = y = 0$". Is the dual statement a theorem of boolean algebra?
Answer:If $x.y = 1$ *then* $x = y = 1$. *This is a theorem of boolean algebra.*