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2-Tree probe interval graphs have a large obstruction set

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Abstract

Probe interval graphs (*PIGs*) are used as a generalization of interval graphs in physical mapping of DNA. $G=(V,E)$ is a *probe interval graph* (*PIG*) with respect to a partition (P,N) of V if vertices of G correspond to intervals on a real line and two vertices are adjacent if and only if their corresponding intervals intersect and at least one of them is in P; vertices belonging to P are called *probes* and vertices belonging to N are called *non-probes*. One common approach to studying the structure of a new family of graphs is to determine if there is a concise family of forbidden induced subgraphs. It has been shown that there are two forbidden induced subgraphs that characterize tree PIGs. In this paper we show that having a concise forbidden induced subgraph characterization does not extend to 2-tree PIGs; in particular, we show that there are at least 62 minimal forbidden induced subgraphs for 2-tree PIGs.

Keywords: Probe interval graphs; 2-Trees; Forbidden induced subgraph characterization

1. Introduction

The *probe interval graph* (*PIG*) model was introduced and used in the human genome project as a more powerful and flexible tool than an interval graph model for the assembly of contigs in the physical mapping of DNA [16–18]. Small fragments of DNA, called clones, are taken from multiple copies of the same genome, and the problem is to reconstruct the

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arrangement of these clones. In other words, physical mapping of DNA has the goal of reconstructing relative positions of clones along the original DNA. This problem of finding whether pairs of clones overlap in a long DNA strand can be modeled by an interval graph if we are interested in overlap information between each pair of clones; vertices represent clones and two vertices are adjacent if and only if the clones overlap. In the PIG model, we can use any subset of clones, called probes, and test for overlap information between a pair of clones if and only if at least one clone is a probe. This allows flexibility, since not all DNA fragments need to be known at the time of the construction of a PIG, as is the case in an interval graph model. Thus, the PIG model can be used in real-time applications with growing data sets by generating incremental DNA maps which provide useful information for each further step. We now give a formal definition of this model.

A graph $G = (V, E)$ is an *intersection graph* of a collection of sets if the vertices of G represent those sets and two distinct vertices are adjacent in G if and only if their corresponding sets have a non-empty intersection. An *interval graph* is an intersection graph of a family of intervals on a real line. $G = (V, E)$ is a PIG with respect to a partition (P, N) of V if vertices of G correspond to intervals on a real line and two vertices are adjacent if and only if their corresponding intervals intersect and at least one of them is in P; vertices belonging to P are called *probes* and vertices belonging to N are called *non-probes*.

There has been a lot of interest in PIGs lately. They have been shown to be weakly triangulated, and thus perfect [12]. The hierarchy of graph classes in the neighborhood of PIGs has been described, and also a new class generalizing chordal graphs to probe chordal graphs has been introduced in analogy to the generalization of interval graphs to PIGs [2,1,7,6]. There exist two recognition problems for PIGs. The first recognition problem asks about recognizing, finding and representing possible layouts of the intervals of a PIG with a given partition of its vertices; we refer to this problem as the *GP recognition problem* (stands for *g*iven *p*artition). The second recognition problem for PIGs asks if a given graph is a PIG without knowing a partition of its vertices; we refer to this problem as the *non-GP recognition problem*. Polynomial time algorithms for the GP recognition problem have recently appeared; in particular, an $O(n^2)$ algorithm [8] and an $O(n + m \log n)$ algorithm [10] have been developed, where n is the number of vertices and m is the number of edges of a graph. An application of an algorithm for constructing a probe interval model occurred in recognizing circular arc graphs [9]. The non-GP recognition problem is unresolved and is attracting considerable attention.

In studying the structure of a new family of graphs a common approach is to determine when the graphs can be characterized by a succinct set of forbidden induced subgraphs. We use the term *FISC* to refer to the forbidden induced subgraph characterization for a family of graphs. In the case of PIGs, as will be seen in the next section, Sheng [15] has taken the first step in this direction by studying FISCs for acyclic PIGs, with or without a given vertex partition. In particular, Sheng solved the non-GP recognition problem for tree PIGs by showing that tree PIGs can be characterized by two forbidden induced subgraphs. This result gives hope that there is a succinct FISC for chordal PIGs, or even PIGs themselves. As a first step in this direction, it is expected that 2-trees, a natural generalization of trees defined in the next section, will have a succinct FISC. Surprisingly, this is not the case. In this paper we show that the FISC for 2-tree PIGs contains at least 62 graphs. Thus, it is very unlikely that there is a concise FISC for PIGs, or even chordal PIGs.

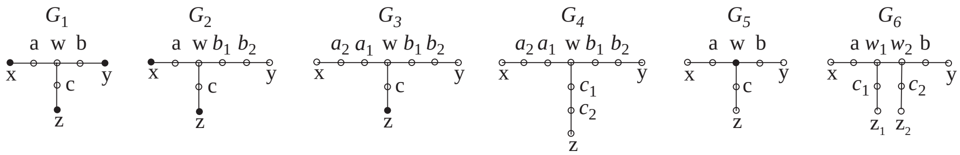

Fig. 1. Forbidden induced subgraphs for tree PIGs, with the darkened vertices being probes and circled vertices being either probes or non-probes.

2. Preliminaries

All graphs in this paper are simple. We denote a graph by $G=(V,E)$, where V is the vertex set of G and E is the edge set of G. We also denote V of G by $V(G)$ and E of G by $E(G)$. For a subset U of V, we denote by $G(U)$ the subgraph of G induced by the vertices of U, and write $G(U) \lhd G$. The standard definitions of *path length* and *path size* are used, representing the number of edges and the number of vertices on the path. The *distance* between vertices u and v in G, denoted by $d(u,v)$, is the number of edges on a shortest uv-path. A graph consisting of a path P_k of size k and a vertex $u \notin V(P_k)$ which is universal to $V(P_k)$ is called a *k-fan*. If K_j is a complete graph on j vertices, a graph G is obtained by K_j*-bonding* of graphs G_1 and G_2 if vertices of a K_j of G_1 are identified with the vertices of a K_j of G_2. The set $N(x)=\{v \in V \mid vx \in E\}$ is the *neighborhood* of vertex x, and $N[x]=N(x)\cup\{x\}$ is the *closed neighborhood* of x. An *asteroidal triple* (*AT*) is an independent set of three vertices in G such that there exists a path between each pair of vertices that avoids the neighborhood of the third vertex. A graph without an AT is called *AT-free*. Vertices of an AT are called *AT-vertices*. We say that a collection of sets $\{X,Y,Z\}$ is an *asteroidal collection* (*AC*) if for all $x \in X$, for all $y \in Y$, and for all $z \in Z$, $\{x,y,z\}$ is an AT. Each of the sets X, Y, and Z is called an *asteroidal set* (*AS*). We defined PIGs in the previous section. An *interval representation*, $I=\{I_v \mid v \in V\}$, of a PIG $G=(V,E_G)$ is a set of intervals of a real line demonstrating that G is a PIG; clearly, the intersection graph $H=(V,E_H)$ of an interval representation I of a PIG G is an interval graph, and $E_G \subseteq E_H$. We now give a recursive definition of a k-tree G: a complete graph on k vertices, K_k, is a *k-tree*; if G is a k-tree, then so is G' formed from G by adding a new vertex adjacent to all vertices in a K_k in G. Thus, a tree is a 1-tree.

As mentioned previously, Sheng has taken the first step in giving a FISC for a restricted family of PIGs, namely trees. In particular, she proved the following [15].

Theorem 1 (*Sheng [15]*). *Let $T=(V,P,E)$ be a tree with $P \subseteq V$ and $N=V\backslash P$. T is a PIG with respect to P if and only if $T(N)$ is an independent set and T has no induced subgraph isomorphic to graph G_i, $i=1,2,\ldots,5$ in Fig. 1, with darkened vertices in P and circled vertices in P or N.*

Theorem 2 (*Sheng [15]*). *Let $T=(V,E)$ be a tree. Then T is a PIG if and only if T has no induced subgraph isomorphic to graph G_4 or G_6 in Fig. 1.*

Lemma 1 (*Sheng [15]*). *At least one AT-vertex of an AT in a PIG must be a non-probe.*

In order to provide the foundation for our search for a FISC for 2-tree PIGs, we now present some general structure results of PIGs.

3. Some structure of PIGs

An immediate consequence of Lemma 1 is the following simple corollary:

Corollary 1. *At least one AS of an AC of a PIG G must contain all non-probes. Thus, at least one AS of a PIG must be an independent set.*

Proof. Otherwise, there exist probe vertices $x \in X$, $y \in Y$, and $z \in Z$ such that $\{x, y, z\}$ is an all probe AT contradicting Lemma 1. □

Claim 1. *Let $\{x, y, z\}$ be an AT of a PIG G. Then in an interval representation of G, no interval corresponding to a vertex in $\{x, y, z\}$ properly contains an interval corresponding to another vertex in $\{x, y, z\}$.*

Proof. Denote by I_x, I_y, and I_z intervals corresponding to x, y, and z, respectively, in an interval representation of G. Without loss of generality, let $I_x \subseteq I_y$. If either x or y are probes, then by the definition of a PIG, they must be adjacent, contradicting $\{x, y, z\}$ being an independent set. Thus, the interesting case is when $x, y \in N$. Let $x, y \in N$. Since $x \in N$, every neighbor of x in G must be a probe. Thus, the neighbor of x on every x, z-path in G must be a probe. Since by the definition of a PIG, the interval corresponding to the neighbor of x on every x, z-path must overlap I_x, and since every neighbor of x in G is a probe, and since $I_x \subseteq I_y$, every neighbor in G of x is adjacent in G to y, and thus y hits every x, z-path in G contradicting $\{x, y, z\}$ being an AT in G. □

Claim 2. *If $\{x, y, z\}$ is an all non-probe AT of a PIG G, with intervals $I_x = [x_1, x_2]$, $I_y = [y_1, y_2]$, and $I_z = [z_1, z_2]$ corresponding to x, y, and z in an interval representation I of G, and if one of these intervals, say I_i, $i \in \{x, y, z\}$, is properly contained in the interval $[a, b]$, where a is the minimum of the left-most vertices and b is the maximum of the right-most vertices of the other two intervals, then there exists a non-probe internal vertex v of a j, k-path such that $I_i \subset I_v$, where $j, k \in \{x, y, z\} \backslash \{i\}$, $j \neq k$.*

Proof. Without loss of generality assume that $I_y \in [x_1, z_2]$. Since the same argument applies to all arrangements of I_x, I_y, and I_z on the real line, we will consider only one of them, namely let $I_x \cap I_y \neq \emptyset$ and let I_z not overlap I_y. Since by Claim 1 no interval in $\{I_x, I_y, I_z\}$ properly contains another, without loss of generality let $x_1 < y_1 < x_2 < y_2$. First, let $y_2 < z_1$, and consider an x, z-path $P_{x,z}$ in G that avoids $N(y)$. Since $x, z \in N$, the neighbor of x in G and the neighbor of z in G on$P_{x,z}$ must both be probe and thus their corresponding intervals cannot overlap I_y. Since the union of the corresponding intervals in I of the vertices of $P_{x,z}$ overlaps I_y, and since $P_{x,z}$ avoids $N(y)$ in G, there must exist a non-probe internal vertex v of $P_{x,z}$ such that $I_v \supset I_y$. Similarly, if $z_1 < y_2$, then there must exist a non-probe internal vertex v on an x, z-path such that $I_v \supset I_y$. □

We now give a structural result on a P, N partition in a PIG with an AT.

Claim 3. *In every AT of a PIG $G = (V, E_G)$ there must exist a non-probe AT vertex u such that there exists a path between the other two AT-vertices that avoids $N(u)$ and has a non-probe internal vertex.*

Proof. Let $I = \{I_v \mid v \in V\}$ be an interval representation of G. Let $H = (V, E_H)$ be the intersection graph of I. Let $\{x, y, z\}$ be an AT of G and without loss of generality let $z \in N$. Since H is an interval graph, H does not have any ATs, so $\{x, y, z\}$ is not an AT of H, and thus we have the following two cases to consider regarding $\{x, y, z\}$ in H: (1) $xy \in E_H$; (2) $xy \notin E_H$ and for every x, y-path $P_{x,y}$ there exists a vertex $u \in V(P_{x,y})$ such that $uz \in E_H$.
(1) First we consider the case when $xy \in E_H$. Since $xy \notin E_G$, this means that $x, y \in N$, and $I_x \cap I_y \neq \emptyset$. Remember also that $z \in N$ by assumption. Thus, by Claim 1, no interval of a vertex in $\{x, y, z\}$ properly contains an interval of another vertex in $\{x, y, z\}$.

Let $I_x = [x_1, x_2]$, $I_y = [y_1, y_2]$, $I_z = [z_1, z_2]$, and since $I_x \cap I_y \neq \emptyset$ and one does not properly contain the other, without loss of generality assume that $x_1 < y_1 < x_2 < y_2$. We now have two cases regarding the position of I_z with respect to I_y.

- First assume that I_z does not overlap I_y. If $y_2 < z_1$, consider an x, z-path $P_{x,z}$ in G that avoids $N(y)$. Here $I_y \subset [x_1, z_2]$, and thus by Claim 2, there exists a non-probe internal vertex on $P_{x,z}$, as required. If $z_2 < y_1$, then $I_x \subset [z_1, y_2]$, and thus by Claim 2, there exists a non-probe internal vertex v of a y, z-path such that $I_x \subset I_v$, as required.
- If I_z overlaps I_y (remember that $x, y, z \in N$, so by Claim 1, $I_z \not\subset I_y$ and $I_y \not\subset I_z$), then we have three possible cases:

  - ◦ If $y_1 < z_1 < y_2 < z_2$, then $I_y \subset [x_1, z_2]$ and thus, by Claim 2, there must exist a non-probe internal vertex v on an x, z-path such that $I_v \supset I_y$, as required.
  - ◦ If $z_1 < y_1 < z_2 < x_2$ (this implies that $z_1 < x_1$, since $I_z \not\subset I_x$), then $I_x \subset [z_1, y_2]$ and thus, by Claim 2, there must exist a non-probe internal vertex v on an y, z-path such that $I_v \supset I_x$, as required.
  - ◦ If $I_x \cap I_y \subset I_z$, then $x_1 < z_1 < y_1 < x_2 < z_2 < y_2$; this is because $I_x \not\subset I_z$ and $I_y \not\subset I_z$ by Claim 1. Now $I_z \subset [x_1, y_2]$ and thus, by Claim 2, there must exist a non-probe internal vertex v on an x, y-path such that $I_v \supset I_z$, as required.

(2) Now consider the case when $xy \notin E_H$ and $uz \in E_H$ for some $u \in V(P_{x,y})$.

- If $u \notin \{x, y\}$, then since $uz \notin E_G$, both u and z are non-probes, and $I_u \cap I_z \neq \emptyset$. Thus, an internal vertex u of $P_{x,y}$ is a non-probe, as required.
- If $u \in \{x, y\}$, without loss of generality let $u = x$, then since $xz \notin E_G$, $I_x \cap I_z \neq \emptyset$ and $x, z \in N$. Since $xy \notin E_H$, $I_x \cap I_y = \emptyset$; y could be a probe or a non-probe. Without loss of generality let $x_1 < x_2 < y_1 < y_2$. By Claim 1, $I_z \not\subset I_x$ and $I_x \not\subset I_z$. Thus we have only two cases to consider: $z_1 < x_1 < z_2 < x_2$ and $x_1 < z_1 < x_2 < z_2$. If $z_1 < x_1 < z_2 < x_2$, then $I_x \subset [z_1, y_2]$ and thus, by Claim 2, there exists an internal non-probe vertex v on a y, z-path such that $I_x \subset I_v$, as required. Similarly, if $x_1 < z_1 < x_2 < z_2$, then $I_z \subset [x_1, y_2]$ and thus, by Claim 2, there exists an internal non-probe vertex v on a x, y-path such that $I_v \supset I_z$, as required. □

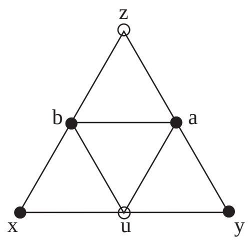

Fig. 2. The only (P, N)-partition of a 3-sun up to isomorphism.

The following is a straightforward corollary of Claim 3.

Corollary 2. *There exists only one* (P, N)*-partition of vertices of a* 3*-sun up to isomorphism.*

Proof. Consider a 3-sun G labeled as in Fig. 2 with the AT-vertices x, y, z. By Lemma 1, at least one of x, y, and z is a non-probe. Without loss of generality, let $z \in N$. Thus, $N(z) = \{a, b\} \subseteq P$. If $x \in N$, then $u \in P$, so all internal vertices of all paths between AT-vertices are probes contradicting Claim 3. Thus, $x \in P$, and similarly, $y \in P$. By Claim 3, $u \in N$. $\square$

4. 2-Tree probe interval graphs

We define a 2-*path* recursively in the following way:

- A triangle, K_3, is a 2-*path* of length one; denote the triangle by t_1.
- $t_0 = \emptyset$.
- If A is a 2-*path* of length k $(k \geqslant 1)$ with the triangle sequence $t_1 t_2 \ldots t_k$, a new length $(k+1)$ 2-*path* is obtained by adding to A a vertex v and edges vv_1 and vv_2, where $v_1 v_2$ is an edge of $t_k \backslash t_{k-1}$; the new triangle induced on $\{v, v_1, v_2\}$ is denoted by t_{k+1}.

An example of a 2-path is presented in Fig. 3. We say that the triangles t_i and t_{i+1}, $1 \leqslant i \leqslant k-1$, of a 2-path A are *consecutive triangles of* A, and that two triangles are *adjacent* if they share an edge. Triangles t_1 and t_k of a length k 2-path are called *end triangles*. A vertex v of degree 2 of an end triangle t_1 or t_k of a 2-path $A = t_1 \ldots t_k$ is called an *end vertex* of A; if $k \geqslant 2$, we denote by v_1 the degree 2 vertex of t_1, and by v_k the degree 2 vertex of t_k. An edge e of an end triangle containing an end vertex is called an *end edge*. An edge of a 2-path A that is not shared between 2 triangles of A and is not an end edge of A is called a *side edge* of A. A non-end, non-side edge of a 2-path A is called an *internal edge* of A. Clearly, the *length* of a 2-path A, denoted by $l(A)$, is the number of triangles in it. Denote by A_i a 2-path of length i. The *distance* between two triangles is the number of edges shared between pairs of consecutive triangles on the shortest 2-path between them.

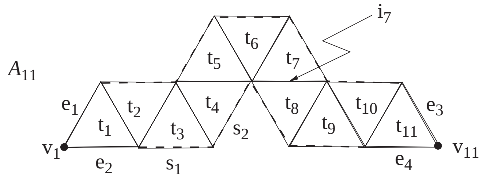

Fig. 3. A 2-path of length 11 with examples of: *end triangles* t_1 and t_{11}, *end vertices* v_1 and v_{11}, *end edges* e_1, e_2, e_3, and e_4, *side edges* s_1 and s_2, and an *internal edge* i_7.

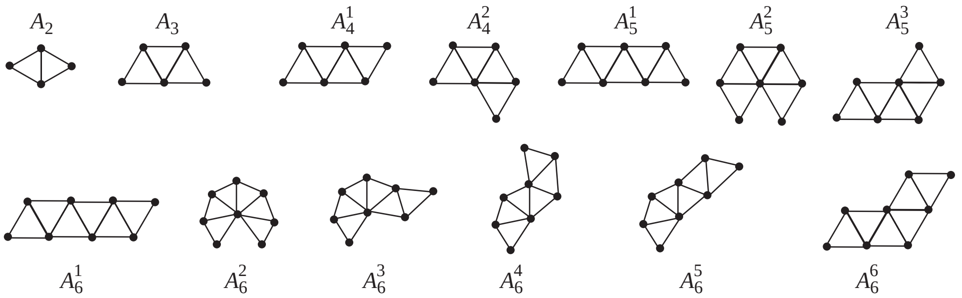

Fig. 4. A_2, A_3, non-isomorphic A_4s, non-isomorphic A_5s, and non-isomorphic A_6s.

Observation 1. *There exists one A_2, one A_3, two non-isomorphic A_4s, three non-isomorphic A_5s, and six non-isomorphic A_6s.*

Proof. By inspection, there are two ways of identifying an edge of an A_3 with an edge of an A_1 to obtain an A_4, three ways of identifying an edge of an A_4 with an edge of an A_1 to obtain an A_5, and six ways of identifying an edge of an A_5 with an edge of an A_1 to obtain an A_6. They are all presented in Fig. 4. □

By identifying a side edge of an A_5 with an end edge of the A_2 in all possible ways so that the resulting 2-tree still has a longest 2-path of length 5, we obtain the two non-isomorphic 2-trees presented in Fig. 5. We call graphs S_1 and S_2 presented in Fig. 5 *weak* 2-*stars*.

Claim 4. *No weak* 2-*star is a PIG.*

Proof. Assume a weak 2-star is a PIG. Consider ACs $\{X, Y, Z\}$ of S_1 and S_2 from Fig. 5, where $X = \{x, x_1\}$, $Y = \{y, y_1\}$, and $Z = \{z, z_1\}$. None of the ASs X, Y, and Z of S_1 and S_2 is an independent set contradicting Corollary 1. □

Consider a 2-path A of length at least 3 of a 2-tree T and denote by v_1v_2 a side edge of A. For a vertex $v \notin V(A)$ of T such that $vv_1, vv_2 \in E(T)$ we say that the triangle vv_1v_2 is an *additional triangle at distance* 1 *from* A and that v is an *additional vertex at distance* 1 *from*

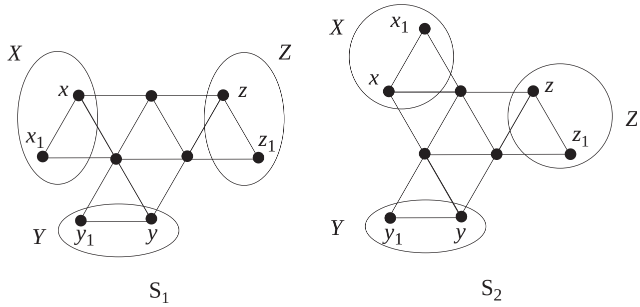

Fig. 5. Weak 2-stars.

A; the number of edges on a shortest 2-path between the triangle vv_1v_2 and a triangle of A is 1. Now consider a 2-path A with an additional triangle vv_1v_2 at distance 1 from A in a 2-tree T. For a vertex $u \notin V(A) \cup \{v\}$ of T such that $uv, uv_i \in E(T)$ for exactly one $i \in \{1, 2\}$, we say that the triangle uvv_i is an *additional triangle at distance* 2 *from A* and that u is an *additional vertex at distance* 2 *from A*; the number of edges on a shortest 2-path between the triangle uvv_i and a triangle of A is 2. Similarly, we can define *additional triangles at distance* 3 *or more from A*. We will use the phrase an *additional triangle with respect to A* to refer to an additional triangle at distance $i \geqslant 1$ from A. When it is clear from the context which A is being considered, we will omit reference to A.

Claim 5. *Let T be a* 2*-tree PIG and let A be a longest* 2*-path of T. T contains no additional triangles at distance* 2 *from A.*

Proof. Assume to the contrary. Let $A = t_1t_2 \ldots t_m$, where $t_1, \ldots, t_m$ are consecutive triangles of A, and let p be an additional triangle at distance 2 from A. Let p be at distance 2 from some t_i of A, and let q be the triangle having an edge in common with p and an edge in common with t_i. Since A is longest, we know that $3 \leqslant i \leqslant m - 2$. But now the subgraph of T induced on the union of the vertices of triangles $t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}, q$, and p is a weak 2-star contradicting T being a PIG, by Claim 4. □

From Claim 5 it follows that if additional triangles with respect to a longest 2-path A of a 2-tree PIG exist, then they must be at distance 1 from A. The next claim determines to which of the P and N vertex partitions of a 2-tree PIG the degree 2 vertices of the additional triangles belong. Recall that we denote by v_1 the degree 2 vertex of t_1, and by v_m the degree 2 vertex of t_m in a 2-path A_m.

Claim 6. *Let T be a* 2*-tree PIG, let $A = t_1t_2 \ldots t_m$ be a longest* 2*-path of T, and let $l(A) \geqslant 4$. Let t be an additional triangle at distance* 1 *from A. Denote by v the degree* 2 *vertex of t.*

- *If t is adjacent to t_2 (or equivalently, to t_{m-1}) and if $v \in P$, then $v_1 \in N$ ($v_m \in N$).*
- *If t is adjacent to t_i for $3 \leqslant i \leqslant m - 2$, then $v \in N$.*

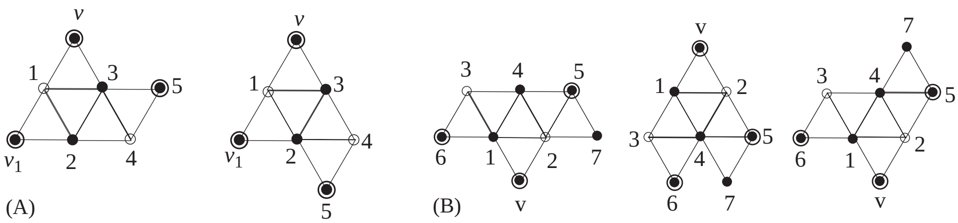

Fig. 6. (A) The two A_4s with an additional triangle. (B) The three A_5s with an additional triangle.

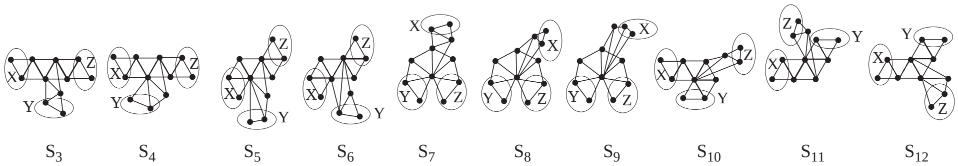

Fig. 7. The ten non-isomorphic 2-stars.

Proof. Assume to the contrary. First, let t be adjacent to t_2 and $v, v_1 \in P$. Let the vertices in $V(t_1) \cup V(t_2) \cup V(t_3) \cup V(t_4) \cup \{v\}$ be labeled as in Fig. 6A, which illustrates the only two non-isomorphic A_4s (by Observation 1) with an additional triangle at distance 1 that is adjacent to t_2 (or equivalently to t_{m-1}). Since the subgraph of T induced on $\{v, v_1, 1, 2, 3, 4\}$ is a 3-sun with the AT $\{v, v_1, 4\}$ and since $v, v_1 \in P$, by Corollary 2, we know that $1, 4 \in N$. Since $4 \in N$ and $45 \in E(T)$, we must have $5 \in P$. But now $\{v, v_1, 5\}$ is an all-probe AT in T contradicting Lemma 1. The proof is the same for the case when t is adjacent to t_{m-1}.

Now assume that t is adjacent to t_i for $3 \leqslant i \leqslant m-2$ and $v \in P$. Clearly, $m \geqslant 5$. Label the vertices of $V(t_{i-2}) \cup V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t_{i+2}) \cup \{v\}$ as in Fig. 6B, which illustrates the only three non-isomorphic A_5s (by Observation 1) with an additional triangle at distance 1 that is adjacent to their t_3s. Clearly, the subgraph of T induced on $\{1, 2, 3, 4, 5, v\}$ is a 3-sun, and since $\{v, 3, 5\}$ is an AT, if we assume that $v \in P$, then by Lemma 1, either $3 \in N$, or $5 \in N$. Without loss of generality, let $3 \in N$. Then by Corollary 2, vertex 2 is also in N. Since $25, 36 \in E(T)$, $5, 6 \in P$. But now we have an all probe AT $\{v, 5, 6\}$ in T contradicting Lemma 1. $\square$

We call a graph obtained by a K_2-bonding of an end edge of an A_3 with the internal edge $e = t_3 \cap t_4$ of an $A_6 = t_1 \dots t_6$ a 2-*star*. There exist two ways of identifying an end edge of an A_3 with the edge $e = t_3 \cap t_4$ of an A_6^1 from Fig. 4 to obtain 2-stars S_3 and S_4 presented in Fig. 7, two ways of identifying an end edge of an A_3 with the edge $e = t_3 \cap t_4$ of an A_6^4 from Fig. 4 to obtain 2-stars S_5 and S_6 presented in Fig. 7, and four ways of identifying an end edge of an A_3 with the edge $e = t_3 \cap t_4$ of each of A_6^2, A_6^3, A_6^5, and A_6^6 from Fig. 4 to obtain sixteen more 2-stars. In this way we constructed twenty 2-stars in total. However, many of them are isomorphic (we tested these isomorphisms manually, as well as by using McKay's *Nauti* 2.0 software [11]). Thus, the following claim holds.

Claim 7. *There exist ten non-isomorphic* 2*-stars. They are denoted by* $S_3, \ldots, S_{12}$ *and presented in Fig. 7.*

Note that S_9 presented in Fig. 7 contains an induced G_4 presented in Fig. 1, which is a non-PIG tree, by Theorem 2. Thus, from now on, we exclude S_9 from the set of 2-stars, since our goal here is to describe non-PIG 2-trees that do not contain non-PIG trees as induced subgraphs.

Claim 8. *No* 2*-star is a PIG.*

Proof. This is because no AS of the AC $\{X, Y, Z\}$ of any of the 2-stars is independent, and thus every 2-star has an all probe AT contradicting Lemma 1. □

Similar to the definition of additional triangles, we now define triangles that "grow off" an internal edge of a longest 2-path of a 2-tree PIG, rather than off a side edge as was the case for additional triangles. Consider a 2-path A of length at least 2 of a 2-tree T and denote by v_1v_2 an internal edge of A. For a vertex $v \notin V(A)$ of T such that $vv_1, vv_2 \in E(T)$ we say that the triangle vv_1v_2 is an *extra triangle at distance* 1 *from* A and that v is an *extra vertex at distance* 1 *from* A; the number of edges on a shortest 2-path between the triangle vv_1v_2 and a triangle of A is one. Now consider a 2-path A with an extra triangle vv_1v_2 at distance 1 from A in a 2-tree T. For a vertex $u \notin V(A) \cup \{v\}$ of T such that $uv, uv_i \in E(T)$ for exactly one $i \in \{1, 2\}$, we say that the triangle uvv_i is an *extra triangle at distance* 2 *from* A and that u is an *extra vertex at distance* 2 *from* A; the number of edges on a shortest 2-path between the triangle uvv_i and a triangle of A is two. Similarly, we define an *extra triangle at distance* 3 *from* A as the triangle wux where $w \notin V(A) \cup \{u, v\}$ and $x \in \{v, v_1\}$, and an *extra vertex* w at distance 3 *from* A. We will use the phrase an *extra triangle with respect to* A to refer to an extra triangle at distance $i \geqslant 1$ from A. When it is clear from the context what A is being considered, we will omit reference to A.

Claim 9. *A* 2*-tree PIG* T *does not contain any extra triangles at distance* 3 *or more from a longest* 2*-path* A *of* T.

Proof. Otherwise, T would contain an induced 2-star contradicting it being a PIG, by Claim 8. In particular, if t is an extra triangle that is at distance 3 from A, and if two shortest 2-paths between t and a triangle of $A = t_1 \ldots t_m$ are $tpqt_i$ and $tpqt_{i+1}$, where t_i and t_{i+1} are two consecutive triangles of A, then we know that $3 \leqslant i < i+1 \leqslant m-2$, since otherwise A would not have been a longest 2-path of T (clearly, $m \geqslant 6$). But now a subgraph of T induced on the vertices in $V(t_{i-2}) \cup V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t_{i+2}) \cup V(t_{i+3}) \cup V(p) \cup V(q) \cup V(t)$ is a 2-star, contradicting T being a PIG, by Claim 8. □

From Claim 9 it follows that if extra triangles with respect to a longest 2-path A of a 2-tree PIG exist, then they must be at distance 1 or 2 from A. The next claim determines the partition to which the degree 2 vertices of the distance 2 extra triangles of A belong. As before, we denote by v_1 the degree 2 vertex of t_1, and by v_m the degree 2 vertex of t_m.

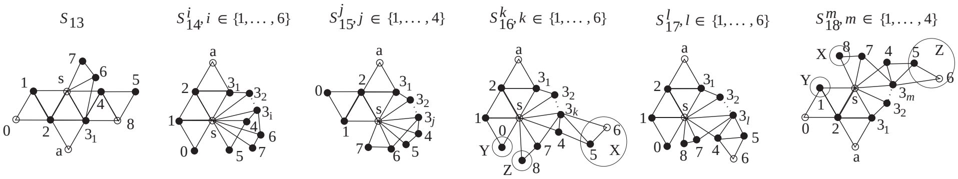

Fig. 8. Twenty-seven minimal forbidden induced subgraphs for 2-tree PIGs resulting from Claim 11.

Claim 10. *Let T be a 2-tree PIG, let $A = t_1 t_2 \dots t_m$ be a longest 2-path of T, and let $m \geqslant 4$. Let t be an extra triangle at distance 2 from A. Denote by v the degree 2 vertex of t. If t is at distance 2 from t_2 and t_3 (or equivalently, from t_{m-1} and t_{m-2}) then*:

- *if $v \in P$, then $v_1 \in N$ ($v_m \in N$);*
- *if $m = 4$ and $v \in P$, then either v_1, or v_m, or both are non-probe.*

If t is at distance 2 from t_i and t_{i+1}, for $3 \leqslant i \leqslant m-3 (m \geqslant 6)$, then $v \in N$.

Proof. If $m = 4$ and if all three vertices v, v_1 and v_4 are probe, than they form an all probe AT in T contradicting Lemma 1. Let $m \geqslant 5$ and let t be at distance 2 from t_2 and t_3 (or equivalently from t_{m-1} and t_{m-2}). Denote by u, w the vertices of $V(t_5) \backslash V(t_3)$ (or vertices of $V(t_{m-4}) \backslash V(t_{m-2})$, if t_{m-1} and t_{m-2} are being considered). Now $\{X, Y, Z\}$ where $X = \{v_1\}$, $Y = \{v\}$, $Z = \{u, w\}$ is an AC, so if $v \in P$, since Z is not independent, v_1 must be a non-probe, by Lemma 1.

We now consider the case when t is at distance 2 from t_i and t_{i+1} for $3 \leqslant i \leqslant m-3 (m \geqslant 6)$. Let u_1, w_1 be the vertices of $V(t_{i-2}) \backslash V(t_i)$, and let u_2, w_2 be the vertices of $V(t_{i+3}) \backslash V(t_{i+1})$. Now $\{X, Y, Z\}$ is an AC, where $X = \{u_1, w_1\}$, $Y = \{u_2, w_2\}$, and $Z = \{v\}$, so since X and Y are not independent, v must be a non-probe, by Lemma 1. □

We now describe some structure of 2-tree PIGs that is forced by the existence of additional and extra triangles with respect to their longest 2-paths.

Claim 11. *Let t' be an additional triangle at distance 1 from a longest 2-path $A = t_1 \dots t_m$ of a 2-tree PIG T, and let t'' be an extra triangle at distance 2 from A. Let t' be adjacent to the triangle t_i and let t'' be at distance 2 from t_i and t_{i+1} of A for $3 \leqslant i \leqslant m-3 (m \geqslant 6)$, let v' be the degree 2 vertex of t', let v'' be the degree 2 vertex of t'', and let s be the vertex in t_i that is not in t'. Then $v''s \notin E(T)$.*

Proof. By Claim 6, $v' \in N$. Thus, $s \in N$, by Corollary 2 applied to the 3-sun induced on vertices of $t_{i-1} \cup t_i \cup t_{i+1} \cup t'$. Also, $v'' \in N$, by Claim 10. Now, since $s, v'' \in N$, they cannot be adjacent. □

Corollary 3. *The twenty-seven 2-trees presented in Fig. 8 are minimal non-PIGs.*

Proof. First we show that none of the graphs in Fig. 8 are PIGs. Notice that vertices *a* and 7 in graphs S_{13}, S^1_{14}, and S^1_{15}, as well as vertices *a* and 8 in graphs S^1_{16}, S^1_{17}, and S^1_{18} violate the conditions of Claim 11 and thus these graphs are not PIGs. We reason about graphs S^i_{14}, S^j_{15}, and S^l_{17} $(i, j, l \geqslant 2)$ as follows. Assume they are PIGs. By Claim 6, $a \in N$, which implies that $s \in N$ by applying Corollary 2 to the 3-sun induced on vertices $\{1, 2, 3_1, 3_2, a, s\}$ in S^i_{14}, S^j_{15} and S^l_{17}. This further implies that all neighbors of s are in P in these graphs, and thus, if $i, j, l \geqslant 3$, the graph induced on vertices $3_n, 3_{n-1}, 3_{n-2}, 4, 5, 6, 7$ in $S^i_{14}(n = i)$ and $S^j_{15}(n = j)$, and vertices $\{3_l, 3_{l-1}, 3_{l-2}, 5, 6, 7, 8\}$ in S^l_{17} is isomorphic to graph G_5 in Fig. 1 with vertex 3_n $(n = i, j, l$ for graphs S^i_{14}, S^j_{15}, and S^l_{17}, respectively) corresponding to vertex w of G_5 and being a probe, contradicting Theorem 1; similarly, if $i, j, l = 2$, then the graph induced on vertices $3_2, 3_1, 2, 4, 5, 6, 7$ in S^2_{14} and S^2_{15}, as well as vertices $3_2, 3_1, 2, 5, 6, 7, 8$ in S^2_{17} is isomorphic to G_5 with $3_2 \in P$ contradicting Theorem 1. In graphs S^k_{16} and S^m_{18} presented in Fig. 8, $a, s \in N$ by Claim 6 and Corollary 2, and thus $\{x, y, z\}$ is an all-probe AT contradicting Lemma 1, where $x \in X$, $y \in Y$, and $z \in Z$.

It is straightforward to verify that the graphs presented in Fig. 8 are non-isomorphic, and that they are minimal non-PIGs (deletion of any vertex from these graphs yields a PIG). □

Note that all graphs presented in Fig. 8 apart from S_{13} contain a fan of some small size. We cannot insert a fan into S_{13} in the same way we did in the other graphs in Fig. 8 for the following reasons. If we insert a k-fan next to vertex 3_1, then S^k_{13} would contain two trees isomorphic to G_5 presented in Fig. 1, one induced on vertices $5, 8, 7, 6, 3_k, 3_{k-1}, 3_{k-2}$ (or in the case that $k = 2$, induced on vertices $5, 8, 7, 6, 3_2, 3_1, a$), and the other induced on vertices $0, 1, s, 3_1, a, 4, 5$. The vertex 3_k in the first copy of G_5 in S^k_{13} corresponds to the vertex w of G_5 in Fig. 1, and so does the vertex s in the second copy of G_5 in S^k_{13}. However, 3_k and s are adjacent in S^k_{13}, which contradicts the condition of Theorem 1 that they both have to be non-probe. No fans larger than the ones indicated in Fig. 8 can be inserted into the other graphs in Fig. 8, since otherwise removal of the vertex $3_{\lceil k/2 \rceil}$ would yield a non-PIG contradicting the minimality of these graphs.

Claim 12. *Let t be an extra triangle at distance* 1 *from a longest* 2*-path* $A = t_1 \ldots t_m$ *of a* 2*-tree PIG T, and let t be adjacent to* t_i *and* t_{i+1} *of A, for* $3 \leqslant i \leqslant m - 3 (m \geqslant 6)$. *There do not exist in T two extra triangles* t'_1 *and* t'_2 *at distance* 2 *from A such that they are adjacent to different edges of t. If* $i = 2$ *or* $i = m - 2$, *then*:

- *exactly one of* t'_1 *and* t'_2 *must have its degree* 2 *vertex a probe*;
- *if in addition* $m = 5$ *and the vertices of T are denoted as in Fig. 9* (*T can be any one of* $T_1, T_2,$ *and* T_3 *in Fig. 9*), *or if* $m = 4$ *and* $T = T_2 \backslash \{7\}$, *where* T_2 *is presented in Fig. 9, then* $v'_1 \in N$ *and* $v'_2 \in P$.

Proof. Assume to the contrary. Denote by v'_1 the degree 2 vertex of t'_1 and by v'_2 the degree 2 vertex of t'_2. First, let $3 \leqslant i \leqslant m - 3 (m \geqslant 6)$. Now the subgraph S of T induced on the union of the vertices of triangles t_i, t, t'_1, and t'_2 is a 3-sun, but by Claim 10 two of its AT-vertices v'_1 and v'_2 are non-probe, contradicting Corollary 2.

Now let $i = 2$ (or equivalently $i = m - 2$). Denote a subset of the vertices of T as in Fig. 9 (T can be any one of T_1, T_2, and T_3). If both v'_1 and v'_2 are non-probes, then the

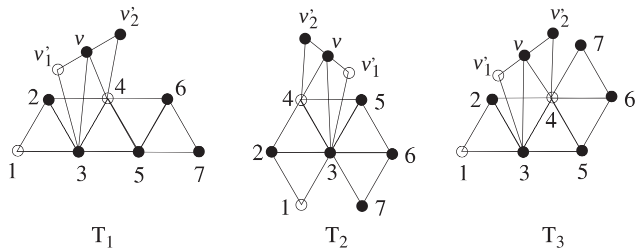

Fig. 9. The three (valid) 2-tree PIGs with two extra triangles at distance 2 from a longest 2-path $A = t_1 \dots t_5$ such that they are adjacent to the same extra triangle.

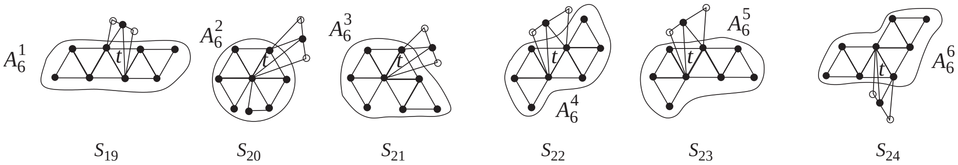

Fig. 10. Six minimal forbidden induced subgraphs for 2-tree PIGs resulting from Claim 12.

3-sun S induced on the set of vertices $\{2, 3, 4, v_1', v, v_2'\}$ has two non-probe AT-vertices contradicting Corollary 2. If both v_1' and v_2' are probes, then by Corollary 2 applied to S, $2 \in N$ and thus $1 \in P$; this contradicts Claim 10 which says that vertex 1 must be a non-probe, since $v_1', v_2' \in P$. Thus, one of v_1' and v_2' must be a probe and the other one non-probe. Note that if $m = 5$, we must have $v_1' \in N$ and $v_2' \in P$, since otherwise we would have $3 \in N$ and thus $1 \in P$, contradicting Claim 10 (another way to see this is: if we would have $v_2' \in N$, $v_1' \in P$, then either $\{1, v_1', 6\}$,or $\{1, v_1, 7\}$, or both would form an all-probe AT contradicting Lemma 1). If $m = 4$, in the 2-tree PIG $T_1\backslash\{7\} = T_3\backslash\{7\}$ presented in Fig. 9 we can have $v_2' \in N$, $v_1' \in P$, in which case $6 \in N$, by Claim 10. However, if $m = 4$ and in $T_2\backslash\{7\}$ we have $v_2' \in N$ and $v_1' \in P$, then $T_2\backslash\{7\}$ would contain an all probe AT $\{v_1', 1, 6\}$ contradicting Lemma 1; thus we must have $v_2' \in P$ and $v_1' \in N$ in $T_2\backslash\{7\}$. □

Corollary 4. *The six* 2*-trees presented in Fig. 10 are minimal non-PIGs.*

Proof. There are six non-isomorphic A_6s, by Observation 1. Using the same notation as in Claim 12, since the "addition" of t, t_1', and t_2' to each of the six non-isomorphic A_6s does not increase the length of the longest path in the resulting graph, we conclude that the six 2-trees presented in Fig. 10 are non-isomorphic. They are non-PIGs, since they violate the conditions of Claim 12. It is easy to see that the removal of any vertex from S_i, $i \in \{19, \dots, 24\}$ yields a PIG, that is, $S_{19}, \dots, S_{24}$ are minimal non-PIGs. □

Claim 13. *Let v_1' and v_2' be the degree* 2 *vertices of two different additional triangles t_1' and t_2' at distance* 1 *from a longest* 2*-path $A = t_1 \dots t_m$ of a* 2*-tree PIG T. Let t_1' be adjacent to t_i and let t_2' be adjacent to t_j of A, $3 \leqslant i < j \leqslant m - 2$, and denote by s_1' and s_2' the vertices in $V(t_i)\backslash V(t_1')$ and $V(t_j)\backslash V(t_2')$, respectively. Then $v_1's_2', v_2's_1' \notin E(T)$.*

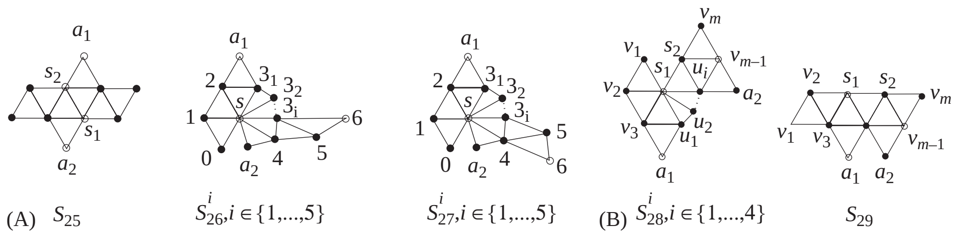

Fig. 11. (A) Eleven minimal forbidden induced subgraphs for 2-tree PIGs resulting from Claim 13. (B) Five minimal forbidden induced subgraphs for 2-tree PIGs resulting from Claim 14.

Proof. If to the contrary, $v'_1 s'_2 \in E(T)$, then at least one of v'_1, s'_2 must be a probe which contradicts the fact that both of them must be non-probe: $v'_1 \in N$ by Claim 6, and $s'_2 \in N$ by Corollary 2 since it belongs to the 3-sun formed by the union of triangles $t_{j-1}, t_j, t_{j+1}, t'_2$, and since $v'_2 \in N$, by Claim 6. $\square$

Corollary 5. *The eleven 2-trees presented in Fig. 11A are minimal non-PIGs.*

Proof. It follows directly from Claim 13 that graphs $S_{25}, S^i_{26}, S^i_{27}, i \in \{1, \ldots, 5\}$ presented in Fig. 11A are not PIGs, since vertices a_1, s_1, a_2, and s_2 in S_{25} as well as vertices a_1, s, a_2, and 3_i in graphs S^i_{26} and S^i_{27} violate the condition described in Claim 13. It is easy to see that these graphs are minimal non-PIGs, since removal of any vertex from any of them yields a PIG. $\square$

Similar to the explanation given after the proof of Corollary 3, no fans can be inserted in the graph S_{25} in Fig. 11A, and no fan larger than a 5-fan can be inserted in the other two graphs in the same figure.

Claim 14. *Let t'_1 and t'_2 be additional triangles at distance one from a longest 2-path $A = t_1 \ldots t_m$ of a 2-tree T that are adjacent to triangles t_i and t_{i+2} of A, respectively, $3 \leqslant i < i+2 \leqslant m-1$, such that there exists a vertex u which satisfies $\{u\} = V(t_i) \cap V(t_{i+2})$ and $\{u\} = V(t'_1) \cap V(t'_2)$. Then T is not a PIG.*

Proof. Denote by a_1 the degree 2 vertex of t'_1, by a_2 the degree 2 vertex of t'_2, by s_1 the vertex in $V(t_i) \backslash V(t'_1)$, and by s_2 the vertex in $V(t_{i+2}) \backslash V(t'_2)$. Clearly, $s_1 s_2 \in E(T)$, by definition of A and u. Assume that T is a PIG. Since t'_1 is adjacent to t_i, $3 \leqslant i \leqslant m-3$, by Claim 6, $a_1 \in N$. Consider the position of the vertex a_2 with respect to A.

- If $i+2 \leqslant m-2$, by Claim 6, we conclude that $a_2 \in N$. In this case, both s_1 and s_2 are non-probe, because the subgraph of T induced on $V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t'_1)$ is a 3-sun with an AT vertex a_1 being a non-probe, and thus by Corollary 2, $s_1 \in N$; similarly, the subgraph of T induced on $V(t_{i+1}) \cup V(t_{i+2}) \cup V(t_{i+3}) \cup V(t'_2)$ is a 3-sun with an AT vertex a_2 being a non-probe, and thus by Corollary 2, $s_2 \in N$. This contradicts s_1 and s_2 being adjacent.
- If $i+2 = m-1$ and $a_2 \in N$, the same argument as above leads to a contradiction. If $a_2 \in P$, then the following argument leads to a contradiction. We know that u, as a

neighbor of $a_1 \in N$, is in P. Thus, since a_2 is also a probe, in the 3-sun induced on $V(t_{m-2}) \cup V(t_{m-1}) \cup V(t_m) \cup V(t'_2)$, by Corollary 2, $v_m \in P$, where v_m is the degree 2 vertex of t_m. Since in the 3-sun induced on the vertices of $V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t'_1)$ we know that $a_1, s_1 \in N$ (the proof is above), this implies that all neighbors of s_1 must be probe. Consider the neighbor $v_{i-1} \in V(t_{i-1}) \setminus V(t_i)$ of s_1. Now vertices $\{a_2, v_m, v_{i-1}\}$ form an all-probe AT in T contradicting Lemma 1. □

Corollary 6. *The five 2-trees presented in Fig. 11B are minimal forbidden induced subgraphs for 2-tree PIGs.*

Proof. The proof that these graphs S^1_{28} and S_{29} are not PIG follows directly from Claim 14. For graphs S^i_{28}, $i \in \{2, \ldots, 4\}$, the proof is similar to the proof of Claim 14: it is easy to see that $a_1, s_1 \in N$ by Claim 6 and Corollary 2, and thus all neighbors of s_1 are probe; also v_{m-1} is a non-probe, by Corollary 2 applied to the 3-sun induced on $\{s_1, s_2, v_m, v_{m-1}, a_2, u_i\}$, since $s_1 \in N$, and thus $v_m, a_2 \in P$ as neighbors of s_1; now $\{v_m, a_2, v_1\}$ form an all probe AT in S^i_{28}, $i \in \{2, \ldots, 4\}$ contradicting Lemma 1.

It is easy to see that the removal of any vertex from any of these graphs makes the resulting graph PIG, that is, these graphs are minimal non-PIG 2-trees. □

Similar to the explanation given after the proof of Corollary 3, no fans can be inserted in the graph S_{29} in Fig. 11B, and no fan larger than a 4-fan can be inserted in the other graph in the same figure.

Combining Theorem 2, Claims 4, 8, and Corollaries 3–6, we have the following:

Theorem 3. *There exist at least 62 graphs in the forbidden induced subgraph characterization for 2-tree PIGs.*

5. Conclusions and future work

We have shown that the FISC for 2-tree PIGs contains at least 62 graphs. It is possible that this list is complete. However, the key point is that this FISC is not concise and thus does not seem to give much insight into the structure of 2-tree PIGs.

It is interesting to note that 13 out of 14 forbidden induced subgraphs for PIGs described in Theorem 2, Claim 4, and Claim 8 have *asteroidal triples of edges*, a structure introduced by Müller [13]: three edges e_1, e_2, and e_3 of a graph G form an asteroidal triple of edges (ATE) if for any two of them there is a path from the vertex set of one to the vertex set of the other that avoids the neighborhood of the third edge, where a neighborhood of an edge $e = uv$ is $N(u) \cup N(v)$. However, the remaining 49 forbidden induced subgraphs for PIGs do not have ATEs and it is not clear if other more general structures occur in these subgraphs. Note that our Corollary 1 is similar to the previously shown result that PIGs cannot have ATEs [3]. Other related results that have appeared recently include a FISC for tree *unit PIGs* [5] and *unit interval bigraphs* [4]; in unit PIGs all intervals in an interval representation of a PIG are of the same length, while unit interval bigraphs are bipartite intersection graphs of two distinct families of the same length intervals with two vertices adjacent if and only if their corresponding intervals overlap and each interval belongs to a distinct family.

Sheng's FISC for tree PIGs [15] implies the existence of an efficient algorithm for solving the non-GP recognition problem for tree PIGs. Using Shamir and Tsur's subtree isomorphism algorithm [14] to determine if each of the two trees in the FISC for tree PIGs (the graphs G_4 and G_6 presented in Fig. 1) is present in a tree T yields an $O(n)$ algorithm for determining if T is a PIG, where n is the number of vertices in T. The problem of efficient non-GP recognition of 2-tree PIGs remains open even if we know a complete FISC for 2-tree PIGs. The more general problems of non-GP recognition of k-tree PIGs for any positive integer k, chordal PIGs, and PIGs in general remain open as well.

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