3004 Computational Complexity problem sheet 2.

1. The set of integers, $\mathbb{Z}$, is defined as containing all the natural numbers along with their negative counterparts (and 0). Formally $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N} \cup \{0\}$, where $-\mathbb{N} = \{-1, -2, -3, ...\}$. Prove that $\mathbb{Z}$ is countable.

2. The set of rational numbers, $\mathbb{Q}$, is defined as the set of all fractions. Formally $\mathbb{Q} = \{\frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \setminus \{0\}\}$. Prove $\mathbb{Q}$ is countable.

3. The set of real numbers, $\mathbb{R}$ is defined informally as being the set of all numbers, including decimals. Examples of real numbers that are not rational include π and $\sqrt{2}$. Prove that $\mathbb{R}$ is not countable (uncountable).

4. Let the alphabet Σ be countably infinite. Prove informally that Σ^* is countable (no need to define a function for this question).

5. Let $\Sigma = \{0, 1\}$ and define $S = \{L : L \text{ is a finite language over } \Sigma\}$. Show S is countable.

6. Recall that the set of all languages $S_{\{0,1\}}$ with the alphabet $\{0, 1\}$ is an uncountable set. Let $C_1, C_2, U \subseteq S_{\{0,1\}}$, where the sets of languages C_1 and C_2 are countable and U is uncountable. The complement of a set A with respect to $S_{\{0,1\}}$ is denoted $\bar{A}$ and is defined as

$$\bar{A} := \{a \in S_{\{0,1\}} : a \notin A\} .$$

   (a) Is the cardinality of $C_1 \cup C_2$,
      i. Countable.
      ii. Uncountable.
      iii. Dependent on choice of C_1 and C_2.

      Give a proof (or detailed explanation) which justifies your answer.

   (b) Is the cardinality of $\bar{C_1}$,
      i. Countable.
      ii. Uncountable.
      iii. Dependent on choice of C_1.

      Give a proof (or detailed explanation) which justifies your answer.

   (c) Is the cardinality of $\bar{U}$,
      i. Countable.
      ii. Uncountable.

iii. Dependent on choice of U.

Give a proof (or detailed explanation) which justifies your answer.

(d) Is the cardinality of $\bar{C}_1 \cap U$,

i. Countable.

ii. Uncountable.

iii. Dependent on choice of C_1 and U.

Give a proof (or detailed explanation) which justifies your answer.

7. Consider the set of all functions $\mathcal{F}$ from the nonnegative integers into the nonnegative integers, i.e., functions f such that $f : \mathbb{N} \to \mathbb{N}$.
Is the cardinality of this set $\mathcal{F}$,

   (a) Countable

   (b) Uncountable

8. Let S be any set and $\mathscr{P}(S)$ be the powerset (set of all subsets) of S. Prove that $|S| < |\mathscr{P}(S)|$, in other words that S is strictly smaller than $\mathscr{P}(S)$. Formally to do this one must show that there can be no function $f : S \to \mathscr{P}(S)$ such that for every $Z \in \mathscr{P}(S)$ there is some $z \in S$ with $f(z) = Z$ (i.e, f is *onto*).