

## Points, Vectors, Lines, Spheres and Matrices

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## Overview

- Points
- Vectors
- Lines
- Spheres
- Matrices
- 3D transformations as matrices
- Homogenous co-ordinates

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## Basic Maths

- In computer graphics we need mathematics both for describing our scenes and also for performing operations on them, such as projection and various transformations.
- Coordinate systems (right- and left-handed), serve as a reference point.
- 3 axes labelled x, y, z at right angles.

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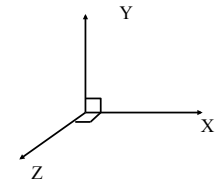
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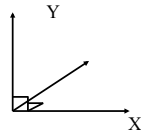
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## Co-ordinate Systems



Right-Handed System  
(Z comes out of the screen)



Left-Handed System  
(Z goes in to the screen)

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## Points, P (x, y, z)

- Gives us a position in relation to the origin of our coordinate system

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## Vectors, V (x, y, z)

- Represent a *direction* (and magnitude) in 3D space
- Points  $\neq$  Vectors  
 $Vector + Vector = Vector$   
 $Point - Point = Vector$   
 $Point + Vector = Point$   
 $Point + Point = ?$

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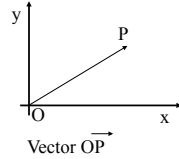
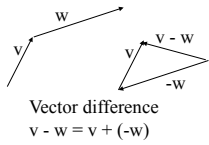
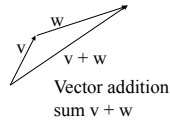
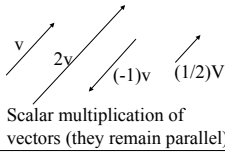
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## Vectors, $V(x, y, z)$




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## Vectors $\underline{V}$

- Length (modulus) of a vector  $\underline{V}(x, y, z)$

$$|\underline{V}| = \sqrt{x^2 + y^2 + z^2}$$

- A unit vector: a vector can be *normalised* such that it retains its direction, but is scaled to have unit length:

$$\hat{V} = \frac{\text{vector } V}{\text{modulus of } V} = \frac{\underline{V}}{|\underline{V}|}$$

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## Dot Product

$$u \cdot v = x_u \cdot x_v + y_u \cdot y_v + z_u \cdot z_v$$

$$u \cdot v = |u| |v| \cos\theta$$

$$\therefore \cos\theta = u \cdot v / |u| |v|$$

- This is purely a scalar number not a vector.
- What happens when the vectors are unit
- What does it mean if dot product == 0 or == 1?

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## Cross Product

- The result is not a scalar but a vector which is normal to the plane of the other 2
- Can be computed using the determinant of:

$$\begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$$\mathbf{u} \times \mathbf{v} = i(y_u z_v - z_u y_v) - j(x_u z_v - z_u x_v) + k(x_u y_v - y_u x_v)$$

- Size is  $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$
- Cross product of vector with itself is null

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## Parametric equation of a line (ray)

Given two points  $P_0 = (x_0, y_0, z_0)$  and  $P_1 = (x_1, y_1, z_1)$  the line passing through them can be expressed as:

$$P(t) = P_0 + t(P_1 - P_0) = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}$$

With  $-\infty < t < \infty$

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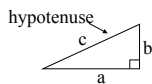
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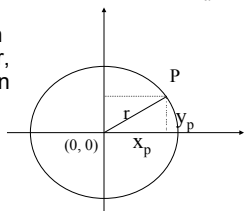
## Equation of a sphere

- Pythagoras Theorem:

$$a^2 + b^2 = c^2$$



- Given a circle through the origin with radius  $r$ , then for any point  $P$  on it we have:



$$x^2 + y^2 = r^2$$

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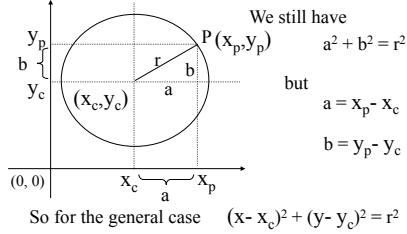
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## Equation of a sphere

\* If the circle is not centred on the origin:




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## Equation of a sphere

\* Pythagoras theorem generalises to 3D giving  
 $a^2 + b^2 + c^2 = d^2$  Based on that we can easily  
 prove that the general equation of a sphere is:

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

and at origin:  $x^2 + y^2 + z^2 = r^2$

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## Matrix Math

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## Vectors and Matrices

- Matrix is an array of numbers with dimensions M (rows) by N (columns)

– 3 by 6 matrix

– element 2,3

is (3)

$$\begin{pmatrix} 3 & 0 & 0 & -2 & 1 & -2 \\ 1 & 1 & 3 & 4 & 1 & -1 \\ -5 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Vector can be considered a 1 x M matrix

–  $v = (x \ y \ z)$

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## Types of Matrix

- Identity matrices - I

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Symmetric

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

- Diagonal

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

– Diagonal matrices are (of course) symmetric

– Identity matrices are (of course) diagonal

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## Operation on Matrices

- Addition

– Done elementwise

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$$

- Transpose

– "Flip" (M by N becomes N by M)

$$\begin{pmatrix} 1 & 4 & 9 \\ 5 & 2 & 8 \\ 6 & 7 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & 6 \\ 4 & 2 & 7 \\ 9 & 8 & 3 \end{pmatrix}$$

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## Operations on Matrices

- **Multiplication**

- Only possible to multiply of dimensions

- $m_1$  by  $n_1$  and  $m_2$  by  $n_2$  iff  $n_1 = m_2$ 
  - resulting matrix is  $m_1$  by  $n_2$
- e.g. Matrix A is 2 by 3 and Matrix B by 3 by 4
  - resulting matrix is 2 by 4
- Just because A x B is possible doesn't mean B x A is possible!

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## Matrix Multiplication Order

- A is  $m$  by  $k$ , B is  $k$  by  $n$
- $C = A \times B$  defined by
 
$$c_{ij} = \sum_{l=1}^k a_{il} b_{lj}$$
- $B \times A$  not necessarily equal to  $A \times B$

$$\begin{pmatrix} * & * & * & * & * \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

$$\begin{pmatrix} * & * & * & * & * \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

$$\begin{pmatrix} * & * & * & * & * \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

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## Example Multiplications

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} - & - \\ - & - \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 3 \\ -3 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$$

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## Inverse

- If  $A \times B = I$  and  $B \times A = I$  then  
 $A = B^{-1}$  and  $B = A^{-1}$

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## 3D Transforms

- In 3-space vectors are transformed by 3 by 3 matrices

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = (xa+yd+zg \quad xb+ye+zh \quad xc+yf+zi)$$

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## Scale

- Scale uses a diagonal matrix

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = (xa \quad yb \quad zc)$$

- Scale by 2 along x and -2 along z

$$\begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = (6 \quad 4 \quad -10)$$

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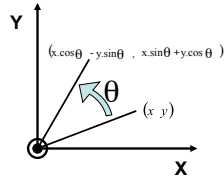
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## Rotation

- Rotation about z axis

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



- Note z values remain the same whilst x and y change

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## Rotation X, Y and Scale

- About X

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

- About Y

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

- Scale (should look familiar)

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

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## Homogenous Points

- Add 1D, but constrain that to be equal to 1 (x,y,z,1)
- Homogeneity means that any point in 3-space can be represented by an infinite variety of homogenous 4D points
  - (2 3 4 1) = (4 6 8 2) = (3 4.5 6 1.5)
- Why?
  - 4D allows us to include 3D translation in matrix form

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## Homogenous Vectors

- Vectors != Points
- Remember points can not be added
- If A and B are points A-B is a vector
- Vectors have form  $(x \ y \ z \ 1)$
- Addition makes sense

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## Translation in Homogenous Form

$$(x \ y \ z \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} = (x+a \ y+b \ z+c \ 1)$$

- Note that the homogenous component is preserved  $(\ast \ast \ast 1)$ , and aside from the translation the matrix is I

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## Putting it Together

$$\begin{pmatrix} R_1 & R_2 & R_3 & 0 \\ R_4 & R_5 & R_6 & 0 \\ R_7 & R_8 & R_9 & 0 \\ T_1 & T_2 & T_3 & 1 \end{pmatrix} = R.T$$

- R is rotation and scale components
- T is translation component

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## Order Matters

- Composition order of transforms matters
  - Remember that basic vectors change so “direction” of translations changed

$$(X \ Y \ Z \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (X \ Z \ -Y \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (X+2 \ Z+3 \ -Y+4 \ 1)$$

$$(X \ Y \ Z \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (X+2 \ Y+3 \ Z+4 \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (X+2 \ Z+4 \ -Y-3 \ 1)$$

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## Exercises

- Calculate the following matrix:  $\Pi/2$  about X then  $\Pi/2$  about Y then  $\Pi/2$  about Z (remember “then” means multiply on the right). What is a simpler form of this matrix?
- Compose the following matrix: translate 2 along X, rotate  $\Pi/2$  about Y, translate -2 along X. Draw a figure with a few points (you will only need 2D) and then its translation under this transformation.

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## Matrix Summary

- Rotation, Scale, Translation
- Composition of transforms
- The homogenous form

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