B-Splines

• Can only achieve $C^1$-continuity
• Need to pay attention to it explicitly!
  – Desirable: basis functions with continuity built-in

Bézier Curves – Joining Curve Segments

• Polynomial curves
• $C^{k-1}$ continuity
  – Cubic B-spline: $C^2$ continuity
• Main idea:
  – Use blossom form as seen with Bézier
  – Instead of using fixed control points (for each segment)
    $f(0,0,0), f(0,0,1), f(0,1,1), f(1,1,1)$
  – Allow “sliding window” of control points, defined by knot vector, e.g.,
    $f(-2,-1,0), f(-1,0,1), f(0,1,2), f(1,2,3)$
    with knot vector $[-2,-1,0,1,2,3]$
Example

- Evaluating Bézier and B-Spline Curves, t in [0,1]

Knots

- A sequence of scalar values \( t_1, \ldots, t_{2k} \) with \( t_i \neq t_j \) if \( i \neq j \), and \( t_i < t_j \) for \( i < j \) (\( k = \text{degree of polyn.} \))
- If \( t_i \) chosen at uniform interval (such as 1,2,3, ...), then it is a uniform knot sequence

Control Points

- We can define a unique k-degree polynomial \( F(t) \) with blossom \( f \), such that \( v_i = f(t_{i+1}, t_{i+2}, \ldots, t_{i+k}) \)
- The sequence of \( v_i \) for \( i \in [0,k] \) are the control points of a B-spline
- Evaluation of a point on a curve with \( f(t,t,t,...) \)
- Remark: no control points will lie on the curve!
Example: Degree 2 (k=2)

• Knots: \( t_1, t_2, t_3, t_4 \)
• Control points
  \[
  v_0 = f(t_1, t_2) = a_0 + a_1 \frac{(t_1+t_2)}{2} + a_2 t_1 t_2
  \]
  \[
  v_1 = f(t_2, t_3) = a_0 + a_1 \frac{(t_2+t_3)}{2} + a_2 t_2 t_3
  \]
  \[
  v_2 = f(t_3, t_4) = a_0 + a_1 \frac{(t_3+t_4)}{2} + a_2 t_3 t_4
  \]

Example: Degree 3 (k=3)

• Knots: \( t_1, t_2, t_3, t_4, t_5, t_6 \)
• Control points
  \[
  v_0 = f(t_1, t_2, t_3)
  \]
  \[
  v_1 = f(t_2, t_3, t_4)
  \]
  \[
  v_2 = f(t_3, t_4, t_5)
  \]
  \[
  v_3 = f(t_4, t_5, t_6)
  \]

Concrete Example for Degree 3

• For the knot sequence \( 1, 2, 3, 4, 5, 6 \)
• Control points
  \[
  v_0 = f(1, 2, 3)
  \]
  \[
  v_1 = f(2, 3, 4)
  \]
  \[
  v_2 = f(3, 4, 5)
  \]
  \[
  v_3 = f(4, 5, 6)
  \]
• Find the point on the curve for \( t = 3.5 \)
Concrete Example for Degree 3

- For \( t = 3.5 \) (\( t \) must lie in \([3,4]\))

\[
\begin{align*}
&f(1,2,3) \quad f(3,4,5) \quad f(2,3,4) \\
&f(3.5,2,3) \quad f(3.4,5) \quad f(4,5,6) \\
&f(3.5,3.5) \quad f(3.5,5,5) \\
&f(3.5,3.5,3) \quad f(3.5,3.5,4) \\
&f(3.5,3.5,3.5)
\end{align*}
\]

Cubic B-Spline Example

- Curve is **not** constrained to pass through any control points

A B-Spline curve is also bounded by the convex hull of its control points.

More than One Segment

- Promised earlier on that there is automatic continuity
- Let's see how this is achieved…
Definition

• Given a sequence of knots, \( t_1, \ldots, t_{2k} \)
• For each interval \([t_i, t_{i+1}]\), there is a \( k \)-th degree parametric curve \( F(t) \) defined with corresponding B-spline control points

\[ v_{ik}, v_{ik+1}, \ldots, v_i \] (sliding window)

Definition

• If \( f() \) is the \( k \)-parameter blossom associated to the curve \( F(t) \) on \([t_i, t_{i+1}]\), then
  – The control points are defined by
    \[ v_j = f(t_{j+1}, \ldots, t_{j+k}) \], \( j = i-k, i-k+1, \ldots, i \)
  – The evaluation of the point on the curve at \( t \in [t_i, t_{i+1}] \) is given by \( F(t) = f(t, t, \ldots, t) \)

– Aside: the \( k \)-th degree Bézier curve corresponding to this curve has the control points:
  \[ P_j = f(t_i, \ldots, t_{i+k}), j = 0, 1, \ldots, k \]

Example (Cubic B-Splines, \( k=3 \))

Knot: \(-2, -1, 0, 1, 2, 3\)

\( t \) in \([0, 1]\):

- \( v_0 = f(-2, -1, 0) \)
- \( v_1 = f(-1, 0, 1) \)
- \( v_2 = f(0, 1, 2) \)
- \( v_3 = f(1, 2, 3) \)

\( t \) in \([1, 2]\):

- \( v_1 = f(-1, 0, 1) \)
- \( v_2 = f(0, 1, 2) \)
- \( v_3 = f(1, 2, 3) \)
- \( v_4 = f(2, 3, 4) \)

\( t \) in \([2, 3]\):

- \( v_2 = f(0, 1, 2) \)
- \( v_3 = f(1, 2, 3) \)
- \( v_4 = f(2, 3, 4) \)
- \( v_5 = f(3, 4, 5) \)

\( t \) in \([3, 4]\):

- \( v_3 = f(1, 2, 3) \)
- \( v_4 = f(2, 3, 4) \)
- \( v_5 = f(3, 4, 5) \)
- \( v_6 = f(4, 5, 6) \)

One Segment       Multiple Segments
Computation – De Boor Algorithm

- Problem: computing a point on the B-spline for $t \in [t_i, t_{i+1}]$

- Recursion formula:

  $$P_i^0(t) = v_i$$
  $$P_i^r(t) = \frac{t-t_i}{t_{i+r}-t_i} P_{i+1}^{r-1}(t) + \frac{t_{i+r+1}-t}{t_{i+r+1}-t_{i+1}} P_{i+1}^{r}(t)$$

- The required point on the curves is $P_{i+1}^r(t)$

Remarks for Cubic B-Splines

- For control points $v_0, v_1, \ldots, v_n$, the required knot sequence is $t_1, t_2, \ldots, t_{n+3}$
- The curve is defined over the range $t \in [t_3, t_{n+1}]$
- There will be $n-2$ curve segments altogether, since each interval $[t_i, t_{i+1}], i=3, 4, \ldots, n$ defines a curve segment
B-Spline Basis

- For a k\textsuperscript{th} degree B-spline
  \[ F(t) = \sum_{i} N_{k,i}(t)v_i \]
- Where the basis functions are
  \[ N_{k,i}(t) = \begin{cases} 
  1 & \text{if } t \in [t_i, t_{i+1}] \\
  0 & \text{otherwise}
  \end{cases} \]
  \[ N_{k,i}(t) = \left( \frac{t-t_i}{t_{i+1}-t_i} \right) N_{k-1,i}(t) + \left( \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} \right) N_{k-1,i+1}(t), \, t \in [t_i, t_{i+1}] \]

B-Spline Basis – Example Degree 1

- For k = 1, \( v_i = f(t_{i+1}) \), \( v_{i+1} = f(t_i) \), and the segment \( t \in [t_i, t_{i+1}] \)
  \[ F(t) = f(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} v_{i+1} + \frac{t - t_i}{t_{i+1} - t_i} v_i \]
- For k = 1, \( v_i = f(t_{i+1}) \), \( v_{i+1} = f(t_{i+2}) \), and the segment \( t \in [t_{i+1}, t_{i+2}] \)
  \[ F(t) = f(t) = \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} v_{i+1} + \frac{t - t_{i+1}}{t_{i+2} - t_{i+1}} v_{i+2} \]

B-Spline Basis – Example Degree 1, contd.

- Then
  \[ N_{k,i}(t) = \left( \frac{t-t_i}{t_{i+1}-t_i} \right) N_{k-1,i}(t) + \left( \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} \right) N_{k-1,i+1}(t), \, t \in [t_i, t_{i+2}] \]
B-Spline Basis Functions – Visually

\( N_0,i \):

\[ Q(t) = \frac{1}{6}R_{i-3} + \frac{3}{6}R_{i-2} + \frac{3}{6}R_{i-1} + \frac{1}{6}R_i \]

B-Spline Basis Functions – Visually

\( N_1,i \):

\[ Q(t) = \frac{1}{4}R_{i-3} + \frac{3}{4}R_{i-2} + \frac{1}{4}R_{i-1} \]

\( N_2,i \):

\[ Q(t) = 0 \]

Cubic B-Splines – Basis Functions Visually

\[ Q(t) = \frac{3}{8}R_{i-3} + \frac{3}{4}R_{i-2} + \frac{1}{8}R_{i-1} \]
Cubic B-Splines – Basis Functions Visually

- Chained together

Knot Insertion

- Inserting new knots in the sequence while maintaining the B-spline curves can be used for
  - Rendering
  - Adding greater flexibility to the curve shape

Multiple Knots

- Duplicating knots can force curve to go through a control point
- Clamped B-Spline goes through start/end point (multiplicity k+1 for start/end knot)

- Example:
  - Cubic B-Spline
  - Knot vector:
    \[0 0 0 1 2 3 3 3\]
Bezier vs. B-Splines

Bezier is not the same as BSpline

• Relationship to the control points is different
B-splines or Bézier curves?

- Bézier curves are B-splines!
- But the control points are different
- You can find the Bézier control points from the B-spline control points
- In the case of a quadratic B-Spline:
  \( P_2 \) is an interpolation between \( v_{i-2} \) and \( v_{i-1} \),
  \( P_1 = v_{i-1} \)
  \( P_2 \) is an interpolation between \( v_{i-1} \) and \( v_i \)

Relation between 3rd-degree Bézier curves and B-Splines

- Constructing Bézier points from B-Spline points

\[
\begin{align*}
& v_{i-3} f(t_{j-2}, t_{j-1}, t_j) \\
& v_{i-1} f(t_j, t_{j+1}, t_{j+2}) \\
& v_{i-2} f(t_{j-1}, t_j, t_{j+1}) \\
& f(t_{j-1}, t_j, t_j) \\
& P_1 f(t_j, t_j, t_{j+1}) \\
& P_0 f(t_j, t_j, t_j) \\
& v_j f(t_{j+1}, t_{j+2}, t_{j+3}) \\
& f(t_{j+1}, t_{j+1}, t_{j+2}) \\
& P_3 f(t_{j+1}, t_{j+1}, t_{j+1}) \\
& P_2 f(t_j, t_{j+1}, t_{j+1})
\end{align*}
\]

Converting between Bézier & B-Spline

original control points as Bézier

new B-Spline control points to match Bézier

new Bézier control points to match B-Spline

original control points as B-Spline
Advantages of B-Splines over Bézier curves

- The convex hull based on m control points is smaller than for Bézier curve
- There is a better local control
- The control points give a better idea of the shape of the curve