Ray Tracing Polyhedra

Overview

- Barycentric Coordinates
- Ray-Polygon Intersection Test
- Affine Transformations

Line Equation

- Recall that given p_1 and p_2 in 3D space, the straight line that passes between is:

$$p(t) = (1-t)p_1 + tp_2$$

for any real number t

- This is a simple example of a **barycentric combination**

Barycentric Combinations

- A barycentric combination is: a weighted sum of points, where the weights sum to 1.
  - Let $p_1, p_2, \ldots, p_n$ be points
  - Let $a_1, a_2, \ldots, a_n$ be weights

$$p = \sum_{i=i}^{n} a_i p_i$$

$$\sum_{i=1}^{n} a_i = 1$$

Implications

- If $p_1, p_2, \ldots, p_n$ are co-planar points then p as defined will be inside the polygon (convex hull) defined by the points, iff

$$0 \leq a_i \quad \forall i$$

- Proof of this is out of scope, but a few diagrams should convince you of the outline of a proof …

Ray-Tracing Polygons

Ray Tracing a Polygon

- Three steps
  - Does the ray intersect the plane of the polygon?
    - i.e. is the ray not orthogonal to the plane normal
  - Intersect ray with plane
  - Test whether intersection point lies within polygon on the plane

Does the ray intersect the plane?

- Ray equation is: $r(t) = p_0 + t*d$
- Plane equation is: $n.(x,y,z) = k$

- Then test is $n.d \mathrel{!}= 0$
  → ray does intersect plane (ray direction and plane are not parallel)

Where does it intersect?

- Substitute line equation into plane equation

$$n \cdot (x_0 + td_x \quad y_0 + td_y \quad z_0 + td_z) = k$$

- Solve for t

$$t = \frac{k - (n \cdot p_0)}{n \cdot d}$$

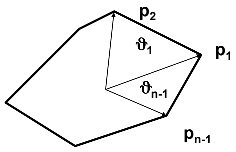
- Intersection: $p_{\text{int}} = p_0 + t * d$

Is this point inside the polygon?

- If it is then it can be represented in barycentric co-ordinates with $0 \le a_i \quad \forall i$
- There are potentially several barycentric combinations (polygon with > 3 vertices)
- Many tests are possible
  - Winding number (can be done in 3D)
  - Infinite ray test (done in 2D)
  - Half-space test (done in 2D for convex poylgons)
  - Barycentric coordinates (in 3D, good for triangles)

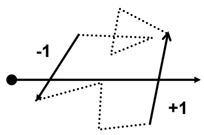
Winding number test

- Sum the angles subtended by the vertices. If sum is zero, then outside. If sum is +/-2Π, inside.
- With non-convex shapes, can get +/-4Π, +/-6Π, etc…



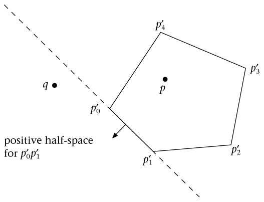
Infinite Ray Test

- Draw a line from the test point to the outside
  - Count +1 if you cross an edge in an anti-clockwise sense
  - Count -1 if you cross and edge in a clockwise sense
- For convex polygons you can just count the number of crossings, ignoring the sign
- If total is even then point is outside, otherwise inside



Half-Space Test (Convex Polygons)

- A point p is inside a polygon if it is in the negative half-space of all the line segments



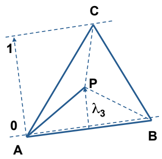
Triangle inside/outside

- Compute barycentric coordinates, and check if all

$$0 \le a_i \quad \forall i$$

- Compute barycentric coords with:
  - λ_1 = Δ(BPC) / Δ(ABC)
  - λ_2 = Δ(APC) / Δ(ABC)
  - λ_3 = Δ(APB) / Δ(ABC)
  - Note: **Δ is signed area**, computed with determinant:

$$\Delta(\mathbf{ABC}) = \frac{1}{2}\begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix}$$



Derivation of BC Computation

- Point P is defined as a barycentric combination:

$$\mathbf{P} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C}$$

- We can write this as a system of linear equations:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ 1 \end{bmatrix}$$

- Solve with Cramer's rule:
  - λ_1 = Δ(BPC) / Δ(ABC)
  - …

Good explanation: http://www.farinhansford.com/dianne/teaching/cse470/materials/BarycentricCoords.pdf

Note

- That the winding angle and half-space tests only tell you **if** the point is inside the polygon, they do not get you a barycentric combination
- With some minor extensions, its easy to show that the infinite ray test finds a barycentric combination.
- Baryc. coord test obviously finds a barycentric combination

Affine Transformations

Transforming Polygons

- Although its sort of "obvious" that transformations of objects preserve flatness and shape, we need to convince ourselves of something specific:
  - **barycentric coordinates are preserved under affine transformations**
- If they weren't it would be extremely hard to shade and texture polygons later on in the course
- **To be shown**: If a transformation is affine (e.g., rotation, scale, translation) then barycentricity is preserved
- If barycentricity is preserved then polygons are still "flat" after transformation

Transformations Revisited

- Homogenous transform *f()* as described is **affine** (by definition, see later)
- Preserves barycentric coordinates iff:

$$f(p) = \sum_{i=1}^{n} \alpha_i f(p_i)$$

$$p = \sum_{i=1}^{n} \alpha_i p_i$$

$$\sum_{i=1}^{n} \alpha_i = 1$$

Show barycentricity is preserved

- Affine Transformation:

$$f(p) = \mathbf{A}p + d$$

  - where A is a (3x3) matrix
  - d is vector

- Or written with a homog. matrix:

$$f(p) = \mathbf{A}p$$

  - where A is a (4x3) matrix as defined earlier

Plug in equations

- Plug in definition of p

$$f(p) = \mathbf{A}\left(\sum_{i=1}^{n} \alpha_i p_i\right) + d$$

$$= \sum_{i=1}^{n} \alpha_i (\mathbf{A}p_i) + d$$

- Remember, want to show:

$$f(p) = \sum_{i=1}^{n} \alpha_i f(p_i)$$

Plug in equations

- Now plug in eq. from other side:

$$\sum_{i=1}^{n} \alpha_i f(p_i) = \sum_{i=1}^{n} \alpha_i (\mathbf{A} p_i + d)$$

$$= \sum_{i=1}^{n} \alpha_i (\mathbf{A} p_i) + \sum_{i=1}^{n} \alpha_i d$$

$$= \sum_{i=1}^{n} \alpha_i (\mathbf{A} p_i) + d$$

$$\Rightarrow \quad f(p) = \sum_{i=1}^{n} \alpha_i f(p_i) \qquad q.e.d.$$

More proofs

- Now, we show that homogenous transformations are actually affine
  (and as such: preserve barycentricity)
- Easy:
  - The transformation $f()$ is exactly the homogenous transformation as defined earlier on. q.e.d.
  - And we already know that $f()$ preserves barycentricity

Recap

- Lines and polygons (and volumes) can be determined in terms of barycentric coordinates
- We have shown that affine transforms preserve barycentricity
- This polygons remain "flat"
- Thus we now can use arbitrary transformations on polyhedra

To Show Transformation is Affine

- Unit vectors e_1=(1 0 0), e_2=(0 1 0), e_3=(0 0 1) and e_4=(0 0 0)
- p=(x_1 x_2 x_3)
- Let x_4=1-x_1-x_2-x_3
- Thus

$$p = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$$

$$p = \sum_{i=1}^{4} x_i e_i$$

…

- But for unit vectors, mapping is easy to derive

$$f(e_i) = (\lambda_{i1} \ \ \lambda_{i2} \ \ \lambda_{i3})$$

$$= \sum_{j=1}^{3} \lambda_{ij} e_j$$

…

- So combine those two

$$f(p) = \sum_{i=1}^{3} x_i \sum_{j=1}^{4} \lambda_{ij} e_j$$

$$= \sum_{j=1}^{3} e_j \sum_{i=1}^{4} \lambda_{ij} x_i$$

…

- But we know $x_4=1-x_1-x_2-x_3$

$$f(p)=\sum_{j=1}^{3} e_j\left(\sum_{i=1}^{4}\lambda_{ij}x_i\right)$$

$$\upsilon_{ij}=\lambda_{ij}-\lambda_{4j}$$

$$f(p)=\sum_{j=1}^{3} e_j\left(\sum_{i=1}^{3}\upsilon_{ij}x_i+\lambda_{4j}\right)$$

…

- Expand that, remembering what e_i is

$$f(p)=\left(\sum_{i=1}^{3}\upsilon_{i1}x_i+\lambda_{41}\quad \sum_{i=1}^{3}\upsilon_{i2}x_i+\lambda_{42}\quad \sum_{i=1}^{3}\upsilon_{i3}x_i+\lambda_{43}\right)$$

- But this is the matrix transformation

$$x'=a_{11}x+a_{21}y+a_{31}z+a_{41}$$

…

Recap

- Lines and polygons (and volumes) can be determined in terms of barycentric coordinates
- We have shown that homog. transforms are affine (and preserve barycentricity)
- This polygons remain "flat"
- Thus we now can use arbitrary transformations on polyhedra