A context-dependent algorithm for merging uncertain information in possibility theory

Anthony Hunter and Weiru Liu

***Abstract*— The need to merge multiple sources of uncertain information is an important issue in many application areas, especially when there is potential for contradictions between sources. Possibility theory offers a flexible framework to represent, and reason with, uncertain information, and there is a range of merging operators, such as the conjunctive and disjunctive operators, for combining information. However, with the proposals to date, the context of the information to be merged is largely ignored during the process of selecting which merging operators to use. To address this shortcoming, in this paper, we propose an adaptive merging algorithm which selects largely partially maximal consistent subsets (LPMCSs) of sources, that can be merged through relaxation of the conjunctive operator, by assessing the coherence of the information in each subset. In this way, a fusion process can integrate both conjunctive and disjunctive operators in a more flexible manner and thereby be more context dependent. A comparison with related merging methods shows how our algorithm can produce a more consensual result.**

***Index Terms*— Information fusion, context-dependent merging, possibility theory, measures of inconsistency and coherence.**

I. INTRODUCTION

Merging multiple sources of uncertain information is an important issue in many areas, such as, sensor data fusion, expert opinion pooling, image data fusion, and multiple classifier results combination (e.g., see a collection of papers on applications in [4]). Different uncertainty modelling theories deploy different combination mechanisms. For example, Dempster's combination rule is used in the Dempster-Shafer theory of evidence, Bayesian's rule is used in probability theory, and disjunctive or conjunctive (or more generally, T-norm and T-conorm) merging rules (operators) are applied in possibility theory. All these rules have certain constraints associated with them. For instance, a conjunctive merging rule is commonly applied to a set of sources which are all reliable and totally agree with each other, otherwise, a disjunctive rule is supposed to be more appropriate. This clear-cut principle for selecting which rule to use was proved to be inadequate for many situations where sources may partially agree with each other and only some sources are reliable. Using a disjunctive rule in these situations often leads to conclusions that are too ignorant to retain much useful information. Attempts have been made to integrate both the conjunctive and disjunctive operators into a single merging procedure to create adaptive rules (e.g. [11], [12], [14], [19]).

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In [11] an adaptive rule was proposed to merge two sources utilizing both conjunctive and disjunctive operators and the degree of agreement among the two sources. Although this rule is one step forward in terms of adaptation, it suffers from two major drawbacks. First, because the rule is not associative, its extension to more than two sources is not obvious. Second, even if it is possible to extend the rule to deal with more than two sources, the rule cannot distinguish a subset of sources where the sources in the subset are totally consistent with each other, whilst all the sources together are conflicting. Therefore, the merge result of these rules cannot truly reflect the nature of information from multiple sources.

In [14], another proposal of adaptation was suggested in which it was assumed that there were j reliable sources among n given sources. Since it was not known which j sources were reliable, all the subsets with cardinality j were considered and sources in each of these subsets were merged conjunctively. The merged results were further merged disjunctively. A method to decide the value of j was given in [12], where j was chosen as the cardinality of the largest subset such that the possibility distribution obtained after conjunctively merging the information in the subset was normal. Although this proposal is better than that in [11] in terms of reflecting the context of information by selecting subsets of sources, it also suffers from a number of drawbacks. Notably, once value j is decided, all the subsets having j as the cardinality are selected for separate conjunctive merges. Most of the selected subsets will contain conflicting information and it does not seem so rational to merge these subsets at all after it is known which subset contains consistent information. Also, merging all these subsets is clearly not feasible when j is large. Another problem with this method is its inability to deal with disjoint consistent subsets of different sizes. For instance, if two consistent subsets among n ($n > 5$) are $S_1 = \{s_1, s_2, s_3\}$ and $S_2 = \{s_4, s_5\}$, then subset S_2 will not be considered since its cardinality is smaller than that of S_1, and value j will be at least equal to $|S_1|$ in this case. Therefore, this rule is not entirely context-dependent either.

In this paper, we further investigate how we can make the merging more context-dependent. We take the merging as a process of finding *largely partially maximal consistent subsets (LPMCSs)* (a concept that we will define later in the paper). We merge information in each LPMCS conjunctively (through relaxation) first and then merge the merged results disjunctively. Although this idea shares some of the spirit of the method in [14], it differs fundamentally from the approach in [14] as to how the subsets should be formed. Our algorithm automatically generates these non-fixed sized subsets instead of fixed sized subsets. Furthermore, it does not require that the conjunctively merged possibility distribution from a LPMCS to be normal. Instead, the definition of relaxation of conjunctive merge (proposed in this paper) is used to assess whether the information in a LPMCS is consistent to at least a certain degree. If so, information in the subset should be merged conjunctively.

More specifically, this paper has the following main contributions to facilitate a context-dependent adaptive merging.

- First, the quality of information provided by each source is assessed and all the sources are ranked. This will enable a potential LPMCS to be created around a source with high quality instead of a source with poor quality. It will also enable the rejection of poor quality sources if necessary.
- Second, the definition of relaxation of the conjunctive rule is proposed which allows the conjunctive merge of sources even though the merged possibility distribution is not normal. This definition assesses both the quality of the conjunctively merged result and the information loss if these sources were merged disjunctively in order to see whether the relaxation is feasible.
- Third, a distance relation is defined which can quantitatively compare the consistency of sources in relation to a reference source. Based on this distance relation, a preferable sequence of sources for merging is created with a given reference source. The relaxation of the disjunctive rule is then applied to the sequence to find the break point such that only the sources before this point can be merged with relaxation. As a consequence, sources before the break point form a largely partially maximal consistent subset (LPMCS).
- Fourth, the merged results from different LPMCSs are merged disjunctively.
- In addition, we have proved that the computational complexity of the algorithm is $O(n^2)$ where n is the number of sources to be merged.

These contributions lead to the design of an adaptive algorithm that dynamically partitions a set of sources based on their context and their consistency with other sources to create LPMCSs. We believe that this algorithm is more context-dependent and can deal with multiple sources involving conflict more adequately than the current approaches available.

This paper is organized as follows. In Section II, we review the basics of possibility theory and introduce the modelling of uncertain information in possibility theory. We also examine problems associated with merging methods that use only a single operator. In Section III, we first introduce both nonspecificity measures and coherence intervals for assessing respectively the quality of consistent and inconsistent possibilistic information and for ranking multiple sources. We then investigate the properties of merged information, including information loss during a disjunctive merge. Definition of relaxation of conjunctive rule is introduced to merge information in a given LPMCS. In Section IV, we study the distance between pieces of possibilistic uncertain information and the formation of LPMCSs. In Section V, an adaptive merging algorithm is designed to find all the LPMCSs and to merge the information in each of these subsets with the relaxation of conjunctive rule. Examples are deployed to illustrate the adaptive behavior of the algorithm. A comprehensive comparison with related adaptive merging approaches are investigated in Section VI and some existing approaches to integrating reliability in merging are reviewed in Section VII. Finally, we summarize the paper in Section VIII.

II. Preliminaries

A. Possibility theory

Let Ω be a frame of discernment consisting of a set of possible solutions. A *possibility measure* and a *necessity measure*, denoted as Π and N respectively, are functions from $\wp(\Omega)$ to $[0,1]$ such that given any two subsets A and B of $\wp(\Omega)$, $\Pi(A \cup B) = \mathsf{max}(\Pi(A), \Pi(B)), N(A) = 1 - \Pi(\bar{A})$, where $\bar{A}$ is the complementary set of A. Also, $\Pi(\Omega) = 1$. The possibility measure of a subset estimates to what extent the true event is believed to be in the subset and its necessity measure evaluates the degree of necessity that the subset is true. Semantically, a fundamental function in possibility theory is a *possibility distribution* $\pi : \Omega \rightarrow [0,1]$. π is said to be *normal* iff $\exists \omega_0 \in \Omega$ such that $\pi(\omega_0) = 1$. A possibility measure Π can be derived from a π as $\Pi(A) = \mathsf{max}(\{\pi(\omega) | \omega \in A\})$. Both functions Π and N are monotonic since $\Pi(A) \leq \Pi(B)$ and $N(A) \leq N(B)$ are true whenever $A \subseteq B$ holds.

Usually, given a set of weighted subsets (A_i, α_i) (may not be all the subsets) of Ω, there is a family of possibility distributions associated with it. Weight α_i is interpreted as the lower bound of a necessity measure on A_i. Therefore, for any possibility distribution π_i that is compatible with this set of weighted subsets, $N_{\pi_i}(A_i) \geq \alpha_i$ must hold where N_{π_i} is the necessity measure associated with π_i.

A common method to select one of the compatible possibility distributions is the *minimum specificity principle* ([2], [9]) which has been widely used in many papers for deriving the least specific possibility distribution that is compatible with (A_i, α_i). This distribution is unique and can be obtained by the following equation.

$$\pi(\omega) = \begin{cases} \mathsf{min}\{1 - \alpha_i | (A_i, \alpha_i) \text{ s.t. } \omega \not\in A_i\} \\ \qquad = 1 - \mathsf{max}\{\alpha_i | (A_i, \alpha_i) \text{ s.t. } \omega \not\in A_i\} \\ 1 \quad \text{otherwise} \end{cases} \tag{1}$$

Let K be the knowledge specifying how some weights are assigned to some subsets of Ω and let the possibility distribution π associated with K be the one obtained through minimum specificity principle in Eq (1). *The degree of inconsistency* of K denoted as $\mathsf{Inc}(K)$ (or simply $\mathsf{Inc}(\pi)$) is defined as

$$\mathsf{Inc}(K) = 1 - \mathsf{max}\{\pi(\omega_i) | \omega_i \in \Omega\}$$

When $\mathsf{Inc}(K) = 0$, π is normal, otherwise, π is not normal.

For any other possibility distribution π_i that is compatible with K, we have $\mathsf{Inc}(\pi_i) \geq \mathsf{Inc}(\pi)$ because π is the least specific possibility distribution. That is the degree of inconsistency of any other compatible possibility distribution is at least to the degree of inconsistency of the least specific possibility distribution we use.

The two basic merging modes in possibility theory are the *conjunctive*, e.g, min, and the *disjunctive* modes, e.g., max. When merging n possibility distributions $(\pi_1, .., \pi_n)$ on Ω with these two specific operators, we have $\forall \omega \in \Omega$,

$$\pi_{cm}(\omega) = \mathsf{min}_i \pi_i(\omega),\ \pi_{dm}(\omega) = \mathsf{max}_i \pi_i(\omega). \quad (2)$$

When using the min operator, the most specific source determines the merged possibility distribution, whilst when using the max operator, the least specific source determines π_{dm}. When a conjunctive merged result is not normal, the disjunctive rule is often recommended to be used. Although there are several merging operators in both the conjunctive and disjunctive modes, in the following, we only consider these two specific merging operators, so the conjunctive operator (or rule, or merge) means using operator min and the disjunctive operator (or rule, or merge) means using operator max.

B. Modelling uncertain information in possibility theory

We discuss uncertain information and the merging of uncertain information in the context of sets in the framework of possibility theory. Assume for each question of interest, there is a collection (called a frame of discernment) of mutually exclusive and exhaustive solutions to the question, then a source can provide information on any subset of the frame. It is usually not possible for a source to specify a degree of belief on every subset. Instead, only some subsets are assigned with degrees of beliefs, either due to partially available knowledge or only some subsets are of interest to the problem concerned. In terms of possibility theory, a degree of belief is interpreted as the lower bound of the degree of necessity of the subset involved.

Let Ω be a frame of discernment consisting of a set of possible solutions. A possibilistic information base (*PIB*) provided by a source (or by an agent's beliefs) is denoted as K and is in the form of a set of weighted subsets

$$K = \{(A_i, \kappa_i)\},\ \ i = 1, .., n$$

where κ_i is interpreted as the lower bound of the necessity degree on subset A_i indicating the belief (from the source) as to what degree the true event (or solution) is in A_i. Given a PIB K, the least specific possibility distribution can be recovered from K using Eq (1). Modelling possibilistic uncertain information in this way is analogous with notations in possibilistic knowledge bases using possibilistic logic (e.g.,[2], [3]), where uncertain knowledge is represented as a set of weighted formulae, $\{(\phi_i, \alpha_i),\ i = 1, \ldots, n\}$. A formula ϕ_i is thought to be equivalent to a subset A_i if ϕ_i is defined as $\phi_i = \vee \psi_j$ where ψ_j stands for "ω_j is true" where $\omega_j \in A_i$.

Although a possibilistic knowledge base can have both (ϕ, α) and (ϕ, β) where $\alpha > \beta$ and possibilistic formula (ϕ, α) subsumes formula (ϕ, β), we restrict our discussion to the case where for each subset in a given PIB, there is only one necessity degree assigned to it. When there are multiple degrees of necessity assigned to the same subset, we only keep the largest value without losing any information.

Example 1 Let $\Omega = \{\omega_1, \omega_2, .., \omega_5\}$ and two PIBs be

$$K_1 = \{(\{\omega_1, \omega_2\}, 0.2), (\{\omega_3, \omega_4\}, 0.3), (\{\omega_5\}, 0.3)\}$$
$$K_2 = \{(\{\omega_1, \omega_2\}, 0.2), (\{\omega_2, \omega_3\}, 0.3), (\{\omega_4, \omega_5\}, 0.3)\}$$

Using Eq (1), the possibility distributions for K_1 and K_2 are

$$\pi_{K_1}(\omega_i) = \pi_{K_2}(\omega_i) = 0.7,\ \ i = 1, ..5$$

which are the same. $\diamond$

When PIBs are represented as collections of weighted subsets, merging information from a pair of sources can be carried out by the following definition.

Definition 1 Let two PIBs be

$$K_1 = \{(A^1_1, \kappa^1_1), .., (A^1_p, \kappa^1_p)\}$$
$$K_2 = \{(A^2_1, \kappa^2_1), .., (A^2_q, \kappa^2_q)\}$$

and let π_1 and π_2 be the corresponding possibility distributions obtained from Eq (1) for K_1 and K_2 respectively. Then the conjunctively merged information K_{cm} using min is

$$K_{cm} = \{(A_1, \kappa_1), .., (A_m, \kappa_m)\}$$

such that

$$A_i \in \{A^1_1, .., A^1_p\} \cup \{A^2_1, .., A^2_q\} \quad (3)$$

$$\kappa_i = 1 - \mathsf{max}(\pi_{cm}(\omega) | \omega \notin A_i) \quad (4)$$

where

$$\pi_{cm}(\omega) = \mathsf{min}(\pi_1(\omega), \pi_2(\omega))$$

The disjunctively merged information K_{dm} using max is

$$K_{dm} = \{(A_1, \kappa_1), .., (A_n, \kappa_n)\}$$

such that

$$A_i \in \{A^1_1, .., A^1_p\} \cup \{A^2_1, .., A^2_q\} \cup \{A^1_i \cup A^2_j\ \ i = 1, .., p, j = 1, .., q\} \quad (5)$$

$$\kappa_i = 1 - \mathsf{max}(\pi_{dm}(\omega) | \omega \notin A_i) \quad (6)$$

where

$$\pi_{dm}(\omega) = \mathsf{max}(\pi_1(\omega), \pi_2(\omega))$$

It is easy to see that both merging procedures are associative and commutative, therefore, each of them can be applied recursively to a set of PIBs. Also in this definition, we assume that an agent is only interested in subsets that are either given in an original PIB or the unions of subsets in PIBs during merging. Therefore, we do not need to exhaustively calculate necessity degrees for all the possible subsets of a frame, though this can be done easily. This will reduce the computation cost considerably. Furthermore, this merging principle is consistent with the policy in merging possibilistic knowledge bases syntactically where only the formulas given in the original knowledge bases are actually involved in merging (e.g., [2]).

Definition 2 Let K_1 and K_2 be two PIBs and let K_{12} be the conjunctively merged PIB. K_1 and K_2 are said to be
(a) *totally consistent* if $\mathsf{Inc}(K_{12}) = 0$;
(b) *partially consistent* if $0 < \mathsf{Inc}(K_{12}) < 1$;
(c) *totally inconsistent* if $\mathsf{Inc}(K_{12}) = 1$.

C. Problems of merging with single operators

To highlight the problem of merging all the information using a single merging rule, we study the following two examples.

Example 2 Consider a set of four PIBs below with $\Omega = \{\omega_1, .., \omega_5\}$.

$K_1^1 = \{(\{\omega_1, \omega_2\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_2\}, 0.4)\}$
$K_2^1 = \{(\{\omega_1, \omega_2\}, 0.3), (\{\omega_1, \omega_2, \omega_3\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$
$K_3^1 = \{(\{\omega_1, \omega_3\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_3\}, 0.4)\}$
$K_4^1 = \{(\{\omega_2, \omega_4\}, 0.3), (\{\omega_1, \omega_3, \omega_4\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$

Let $\pi_1^1, \pi_2^1, \pi_3^1$ and π_4^1 be the corresponding possibility distributions of these PIBs as detailed in Table 1.

TABLE I
FOUR POSSIBILITY DISTRIBUTIONS FOR THE FOUR PIBS.

PIB	π	ω_1	ω_2	ω_3	ω_4	ω_5
K_1^1	π_1^1	0.5	1.0	0.6	0.6	0.5
K_2^1	π_2^1	1.0	0.6	0.6	0.5	0.5
K_3^1	π_3^1	0.5	0.6	1.0	0.6	0.5
K_4^1	π_4^1	0.7	0.5	0.6	1.0	0.5

Since the conjunctively merged result shows that $\pi_{cm}^1(\omega_i) < 1.0$ for all $\omega_i \in \Omega$, it indicates these four sources cannot be merged using the conjunctive rule. When combined with the disjunctive operator, the merged possibility distribution is

$$\pi_{dm}^1(\omega_i) = 1, i = 1, 2, 3, 4, \pi_{dm}^1(\omega_5) = 0.5$$

which suggests that excessive use of the disjunctive rule can result in a merged possibility distribution with little information retained. Therefore, neither of the merges with a single rule is adequate. $\diamond$

Example 3 Consider another set of four PIBs with $\Omega = \{\omega_1, .., \omega_5\}$. Their corresponding possibility distributions are given in Table II.

$K_1^2 = \{(\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_2\}, 0.4)\}$
$K_2^2 = \{(\{\omega_1\}, 0.3), (\{\omega_1, \omega_3\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$
$K_3^2 = \{(\{\omega_2, \omega_3\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5)\}$
$K_4^2 = \{(\{\omega_1, \omega_2\}, 0.3), (\{\omega_1, \omega_3, \omega_4\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$

TABLE II
FOUR POSSIBILITY DISTRIBUTIONS FOR THE FOUR PIBS IN EXAMPLE 3.

PIB	π	ω_1	ω_2	ω_3	ω_4	ω_5
K_1^2	π_1^2	0.5	1.0	0.6	0.6	0.5
K_2^2	π_2^2	1.0	0.5	0.6	0.5	0.5
K_3^2	π_3^2	0.5	1.0	1.0	0.6	0.5
K_4^2	π_4^2	1.0	0.5	0.6	0.7	0.5

Similar to Example 2 that the only merging rule applicable to these PIBs would be the disjunctive one. However, it differs from Example 2 in that two pairs of PIBs in the example, (K_1^2, K_3^2) and (K_2^2, K_4^2) can be merged conjunctively, since the two PIBs in each pair are totally consistent. $\diamond$

Merging all the four sources in the disjunctive mode ignores the fact that some subsets of sources are totally consistent, although the whole collection of all the sources is partially consistent. This situation cannot be reflected in a merge that uses only a single operator. Therefore, a selective use of the conjunctive rule may still be necessary, especially when some subsets are totally consistent. As a consequence, developing methods for measuring the quality of merged results of selected subsets are crucial.

III. QUALITY MEASURES OF ORIGINAL AND MERGED UNCERTAIN INFORMATION

In this section, we investigate how to measure the quality of uncertain information from a single source and the quality of merged results.

A. Quality measure of individual uncertain information

There are a number of dimensions that we can look into when assessing the quality of information from a source, each of which is suitable for certain situations. For information that is consistent, we use the traditional information theory based method to examine its nonspecificity; for information that is inconsistent, we examine its degree of inconsistency. When information from two sources have the same degree of inconsistency, we need to measure their coherence intervals [18] to assess which source is more coherent. Based on all these measures, we are able to rank a set of sources according to their quality. We look at each of these measures in turn below first before examining the quality of merged information.

1) Measures of nonspecificity: When a piece of information is uncertain, e.g., when the information cannot precisely reveal what the true value (of a question) is, numerous methods have been proposed to quantify how uncertain this information actually is. The first of all would be Hartley's approach [16] where the uncertainty is measured by $I(A) = log_2|A|$. Here $A \subseteq \Omega, A \neq \emptyset$ represents the smallest subset of Ω (a frame) that it is certain that the true value (state) is in A. For a probability distribution p on Ω, the Shannon entropy [21] defined below is commonly applied.

$$S(p) = -\Sigma_{\omega \in \Omega} p(\omega) log_2 p(\omega)$$

This method was extended in [17] to measure the uncertainty in information in possibility theory, called *nonspecificity*. Given a possibility measure π on Ω with $\pi(\omega_1) = 1 \geq \pi(\omega_2) \geq .. \geq \pi(\omega_n) = 0$, the measure of nonspecificity of π is

$$H(\pi) = \Sigma_{j=1}^{n-1} (\pi(\omega_j) - \pi(\omega_{j+1})) log_2 j$$

When a π is precise, i.e., when there is only one $\omega_i \in \Omega$ where $\pi(\omega_i) = 1$ and $\pi(\omega_j) = 0, i \neq j$, then $H(\pi) = 0$. When a π is uniform, i.e., when $\forall \omega \in \Omega, \pi(\omega) = 1$, then $H(\pi) = log_2 n$. This represents total ignorance.

The measure of nonspecificity defined above is usually used when the possibility measure is normal.

Note that the above definition is valid only when $min_{\omega \in \Omega} \pi(\omega) = 0$. When this condition does not hold, we need to add an extra element ω' into Ω and let $\pi(\omega') = 0$ in order to use the equation. It should also be noted that $H(\pi) = 0$ whenever $\pi(\omega_1) = l$ and $\pi(\omega_i) = 0$ for any $\omega_i \neq \omega_1$, regardless the actual value of l. This raises a question as whether $\pi_1(\omega_1) = 1$ should be treated the same as $\pi_2(\omega_1) = 0.8$ (or any other values, such as 0.2), because we have $H(\pi_1) = H(\pi_2) = 0$. We argue on the one hand that if for a given π, $\pi(\omega_1)$ is very small, e.g., 0.2, then it is unlikely that $\pi(\omega_i) = 0$ holds for all the other ω_i, so most of the time, $\pi(\omega_1)$ cannot be too small. On the other hand, if $\pi(\omega_1)$ is the only non-zero value and it is reasonably large, then this possibility distribution is specific enough, since degrees of possibility can be viewed as relative measures after all. Therefore, a possibility distribution π with $\pi(\omega_1) = 0.8$ (and $\pi(\omega_i) = 0$ for $\omega_i \neq \omega_1$) can be regarded as carrying the same information as a possibility distribution $\pi'(\omega_1) = 1$ (with $\pi'(\omega_i) = 0$ for $\omega_i \neq \omega_1$).

2) Inconsistency degree and coherence interval: An inconsistency degree can tell to what extend a PIB is internally consistent. In Example 1, the degrees of inconsistencies of the two PIBs are the same, $1 - \mathsf{max}_{\omega\in\Omega}\{\pi_1(\omega)\} = 0.3$ and $1 - \mathsf{max}_{\omega\in\Omega}\{\pi_2(\omega)\} = 0.3$. Therefore, it is not possible to tell the difference of the quality of these two sources based on this value. However, if we examine the information more closely, we will find that there is a significant overlap among the subsets A_j^2 in K_2, while any two subsets from K_1 are disjoint. Obviously the quality of K_2 is better than that of K_1, since the latter is more conflicting internally. This observation leads to the study of coherence intervals of subsets within a single piece of information [18], which extends the work in [8].

Definition 3 [18] Let K be a PIB and let

$$\begin{aligned}&\mathsf{OpinionBase}(K) = \{(A_i, \kappa_i)|\\ &\qquad \text{such that } (A_i,\kappa_i) \text{ is a weighted subset of K } \}\end{aligned}$$

$$\begin{aligned}&\mathsf{ConflictBase}(K) = \{(A_i, \kappa_i) \in \mathsf{OpinionBase}(K)|\\ &\qquad \exists (A_j,\kappa_j) \in \mathsf{OpinionBase}(K), \quad \text{s.t } A_i \cap A_j = \emptyset\}\end{aligned}$$

$$\begin{aligned}&\mathsf{UpperConflictBase}(K) = \{(A_i, \kappa_i) \in \mathsf{OpinionBase}(K)|\\ &\qquad \forall (A_j,\kappa_j) \in \mathsf{OpinionBase}(K),\\ &\qquad\quad \text{if } A_i \neq A_j \text{ then } \; A_i \cap A_j = \emptyset\}\end{aligned}$$

Then the *lower bound of the degree of coherence* of K, denoted as $\mathsf{Coherence}(K)$, and the *upper bound of the degree of coherence* of K, denoted as $\mathsf{UpperCoherence}(K)$, are defined as follows where $A(S) = \Sigma_{(A_i,\kappa_i)\in S} \;\; \kappa_i$

$$\mathsf{Coherence}(K) = 1 - \frac{A(\mathsf{ConflictBase}(K))}{A(\mathsf{OpinionBase}(K))}$$

$$\mathsf{UpperCoherence}(K) = 1 - \frac{A(\mathsf{UpperConflictBase}(K))}{A(\mathsf{OpinionBase}(K))}$$

Interval $[\mathsf{Coherence}(K), \mathsf{UpperCoherence}(K)]$ defines the range of coherence measure of K with some special properties, such as, $[1,1]$ means K is totally coherent, while $[0,0]$ means K is totally incoherent [18]. When the possibility distribution derived from a source is normal, the associated coherence interval is $[1,1]$. However, a $[1,1]$ coherence interval does not guarantee that the underlying possibility distribution is normal.

Definition 4 [18] Let $\leq_{lex}$ be a binary relation on the following set where $1 > \alpha > 0$ and $1 > \beta > 0$,

$$\{[1,1],[\alpha,1],[0,1],[0,\beta],[0,0]\}$$

such that $[0,0] \leq_{lex} [0,\beta]$; $[0,\beta] \leq_{lex} [0,1]$; $[0,1] \leq_{lex} [\alpha,1]$; $[\alpha,1] \leq_{lex} [1,1]$; $[\alpha_1,1] \leq_{lex} [\alpha_2,1]$ if $\alpha_1 \leq \alpha_2$; and $[0,\beta_1] \leq [0,\beta_2]$ if $\beta_1 \leq \beta_2$. $\leq_{lex}$ is a lex ordering and its associated strict ordering is defined by $[\alpha_1,\alpha_2] <_{lex} [\beta_1,\beta_2]$ such that $[\alpha_1,\alpha_2] \leq_{lex} [\beta_1,\beta_2]$ but $[\beta_1,\beta_2] \not\leq_{lex} [\alpha_1,\alpha_2]$.

Let K_1 and K_2 be the PIBs in Example 1 respectively, their coherence intervals are $I_{K_1} = [0,0]$ and $I_{K_2} = [0,5/8]$ and $I_{K_1} <_{lex} I_{K_2}$. This means that the quality of K_2 is better than that of K_1 which is consistent with our analysis above.

3) Ranking information sources: The quality of information provided by a source can be measured in several aspects, as we have discussed above. In general, when the information is consistent, it is possible to measure its nonspecificity; when it is inconsistent, it is meaningful to measure its inconsistency degree and its coherence interval. Given K_1 and K_2 with the same degree of inconsistency, their coherence intervals can further differentiate a better quality information source from a poor one. Equipped with these measures, we can rank a collection of PIBs.

Definition 5 Let K_1 and K_2 be two PIBs. The quality of K_1 is said to be *better than* that of K_2, denoted as $K_2 \prec K_1$ iff one of the following conditions holds.
(a) K_1 and K_2 are consistent and $H(\pi_{K_1}) < H(\pi_{K_2})$;
(b) $\mathsf{Inc}(K_1) < \mathsf{Inc}(K_2)$;
(c) $\mathsf{Inc}(K_1) = \mathsf{Inc}(K_2)$ and $I_{K_2} <_{lex} I_{K_1}$, where I_{K_1} and I_{K_2} are the coherence intervals for K_1 and K_2.

In addition, notation $K_1 \sim K_2$ means their qualities are not distinguishable; and notation $K_2 \preceq K_1$ means the quality of K_1 is *better than or equal to* that of K_2.

Clearly, relation $\preceq$ is transitive, but not associative. If $K_i \preceq K_j, K_j \preceq K_t$, then $K_i \preceq K_t$.

This definition ranks consistent information ahead of inconsistent information and a highly specific consistent information is better than a less specific consistent information. Furthermore, information with a lower degree of inconsistency is better than information with a higher one. When two pieces of information have the same degree of inconsistency, their coherence intervals are used to identify which information is better.

Example 4 The PIBs in Example 2 can be ordered as
$K_1^1 \sim K_2^1, K_2^1 \sim K_3^1$, and $K_4^1 \prec K_3^1$
since $H(\pi_1^1) = H(\pi_2^1) = H(\pi_3^1) = 0.1log_2 3 + 0.5log_2 5$ and $H(\pi_4^1) = 0.1log_2 2 + 0.1log_2 3 + 0.5log_2 5$, which gives $H(\pi_i^1) < H(\pi_4^1)$ for $i = 1,2,3$. Therefore, K_4^1 is of poor quality than others. $\diamond$

B. Quality measure of merged information

1) Properties of merged information: Since operators min and max are both associative and commutative, the merging procedure defined in Definition 1 can be applied to a set of PIBs recursively until all the information is combined, provided that there is no normalization involved in the merge. This process should achieve the same result as that from merging all the information in a single step using Eq (2). To quantitatively measure the effects of both types of merging, we have the following propositions.

Proposition 1: Let $K_i, i = 1, ..., n$ be n PIBs and let (A_i^j, κ_i^j) be a weighted subset in K_i, then (A_i^j, κ) is a weighted subset in the conjunctively merged K_{cm} such that $\kappa_i^j \leq \kappa$.

Proof Let π_i and π_{cm} be the possibility distributions associated with K_i and K_{cm} respectively. Then

$$\kappa_i^j = 1 - \Pi_i(\bar{A}_i^j) = 1 - \mathsf{max}(\pi_i(\omega_l)|\omega_l \in \bar{A}_i^j)$$
$$\kappa = 1 - \Pi_{cm}(\bar{A}_i^j) = 1 - \mathsf{max}(\pi_{cm}(\omega_l)|\omega_l \in \bar{A}_i^j)$$

Since $\pi_{cm}(\omega_l) \leq \pi_i(\omega_l)$ for $\omega_l \in \bar{A}_i^j$ (remember $\pi_{cm}(\omega_l) = \mathsf{min}(\pi_t(\omega_l))$), $\mathsf{max}(\pi_i(\omega_l)) \geq \mathsf{max}(\pi_{cm}(\omega_l))$. Therefore, $1 - \mathsf{max}(\pi_i(\omega_l)|\omega_l \in \bar{A}_i^j) \leq 1 - \mathsf{max}(\pi_{cm}(\omega_l)|\omega_l \in \bar{A}_i^j)$, and as a consequence, $\kappa_i^j \leq \kappa$. $\diamond$

Proposition 2: Let $K_i, i = 1, ..., n$ be n PIBs and let subset A be a weighted subset in l different PIBs with (A, κ_i^l) appearing in K_i. Then (A, κ) is a weighted subset in the conjunctively merged K_{cm} such that $\kappa \geq \mathsf{max}\{\kappa_i^l|(A, \kappa_i^l) \in K_i\}$.

Proposition 2 can be proved similarly as for Proposition 1.

Proposition 3: Let $K_i, i = 1, ..., n$ be n PIBs and let subset A be a weighted subset appearing in l different PIBs with (A, κ_i^l) appearing in K_i. If (A, κ) is a weighted subset in the disjunctively merged K_{dm}, then $\kappa \leq \mathsf{min}(\kappa_i^l|(A, \kappa_i^l) \in K_i)$.

Proof Let π_s and π_{dm} be the possibility distributions associated with K_i and K_{dm} respectively. Let (A, κ_s^l) be a weighted subset in K_s (one of the l PIBs having A as a weighted subset) and assume (A, κ) appears in the merged K_{dm}, then

$$\kappa_s^l = 1 - \Pi_i(\bar{A}) = 1 - \mathsf{max}(\pi_s(\omega_l)|\omega_l \in \bar{A})$$
$$\kappa = 1 - \Pi_{dm}(\bar{A}) = 1 - \mathsf{max}(\pi_{dm}(\omega_l)|\omega_l \in \bar{A})$$

Since $\pi_{dm}(\omega_j) \geq \pi_s(\omega_j)$ for $\omega_j \in \bar{A}$ (remember $\pi_{dm}(\omega_j) = \mathsf{max}_t(\pi_t(\omega_j))$), we have $\mathsf{max}_j(\pi_{dm}(\omega_j)) \geq \mathsf{max}_j(\pi_s(\omega_j))$. Therefore, $1 - \mathsf{max}(\pi_s(\omega_j)|\omega_j \in \bar{A}) \geq 1 - \mathsf{max}(\pi_{dm}(\omega_j)|\omega_j \in \bar{A})$, and as a consequence, $\kappa_s^l \geq \kappa$. That is $\kappa \leq \kappa_s^l$.

Similarly, when (A, κ_t^l) is a weighted subset in K_t and (A, κ) is a weighted subset in K_{dm}, we also have $\kappa \leq \kappa_t^l$.

Therefore, $\kappa \leq \mathsf{min}(\kappa_i^l|(A, \kappa_i^l) \in K_i)$. $\diamond$

These propositions reveal that any weighted subset in an original source will not disappear during the conjunctive merge. Furthermore, if a subset appears in more than one source (with different weights), the conjunctively merged weight associated with the subset is not less than the original weights. On the other hand, any weighted subset, (A_i, κ_i), that appears in K_i will be deleted from the merged K if (A_i, κ) has $\kappa = 0.0$ after merging. Since the disjunctive merge can result in some subsets having 0.0 degree of necessity, not all the subsets that appear in the original sources will be in the merged result. This leads to the loss of some of the original information.

2) Information loss in disjunctive merge: With each more step of disjunctive merge, the measure of nonspecificity of the merged result gets bigger and the merged information gets more imprecise. That is, $\mathsf{max}(H(\pi_1), H(\pi_2)) \leq H(\pi_{dm})$ holds in general, where π_1 and π_2 are two normal possibility distributions from two sources and π_{dm} is the disjunctively merged possibility distribution. To measure how much of the original uncertain information has been lost during merging, we give the following definition.

Definition 6 Let $K_i, i = 1, ..., n$ be n PIBs. Let $S_i = \{A_i|(A_i, \kappa_j) \in K_i\}$ be the collection of subsets appearing in K_i, and let $S_{dm} = \{A|(A, \kappa) \in K_{dm}\}$ be the collection of subsets appearing in the disjunctively merged K_{dm}. *Information loss* in relation to (A_i, κ_i) is defined as $A_i \notin S_{dm}$. The *degree of information loss* of the disjunctive merge is defined as

$$\mathsf{InfoLoss}(K) = 1 - \frac{|S_{dm} \cap (\cup_i S_i)|}{|\cup_i S_i|} \qquad (7)$$

In Example 2, after disjunctively merging all the four PIBs in the first set, the degree of information loss is 1.0. That is, none of the original subsets was kept.

3) Relaxation of the conjunctive rule: When the information loss is too significant to apply the disjunctive rule, the conjunctively rule has to be applied in some way. We propose the following definition to suggest which rule should be applied based on the assessment of conjunctive and disjunctive merges of a set of PIBs.

Definition 7 Let $K_1, K_2, .., K_n$ be n PIBs and let K_{cm} and K_{dm} be the conjunctively and disjunctively merged PIBs respectively. Then these PIBs

(a) *should be merged conjunctively* when $\mathsf{Inc}(K_{cm}) = 0$;
(b) *are advised to be merged conjunctively* when

(i) $0 < \mathsf{Inc}(K_{cm}) \leq \epsilon_0$;
(ii) $I_{K_{cm}} = [\alpha, 1]$ such that $\alpha \geq \epsilon_1$;
(iii) $\mathsf{InfoLoss}(K_{dm}) \geq \epsilon_2$;

(c) *are advised to be merged disjunctively*, otherwise.

ϵ_0, ϵ_1 and ϵ_2 are three pre-defined thresholds such that ϵ_0 is the degree of inconsistency tolerance, ϵ_1 measures the least level of coherence the merged result should achieve, and ϵ_2 is the threshold for the degree of information loss.

The application of the conjunctive rule in situation (b) above is called the *relaxation of the conjunctive rule*, or simply called *relaxation* in the rest of the paper.

In general, the closer ϵ_1 is to 1.0, the more coherent the PIBs are, however, the closer ϵ_2 is to 1.0, the more severe the information loss is. Some applications may tolerate a high degree of inconsistency but require a higher coherence interval then other applications. Therefore, these thresholds should be tuned according to specific applications.

The relaxation of the conjunctive rule deals with those situations where although the conjunctively merged result of a set of PIBs is not normal, nevertheless, its inconsistent degree is low, it is highly coherent, and if these PIBs are merged disjunctively its information loss is too significant. Possibilistic information bases (PIBs) that can be merged with the relaxation of the conjunctive rule are said to be *largely partially consistent*.

Example 5 Let two more sets of PIBs, which are numbered as the 3rd and the 4th sets, be given as below

$$K_1^3 = \{(\{\omega_1, \omega_3\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_2, \omega_3\}, 0.4)\}$$
$$K_2^3 = \{(\{\omega_1, \omega_2\}, 0.3), (\{\omega_1, \omega_2, \omega_3\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$$
$$K_3^3 = \{(\{\omega_1, \omega_2, \omega_3\}, 0.4), (\{\omega_1, \omega_2, \omega_4\}, 0.4), (\{\omega_2, \omega_3\}, 0.4)\}$$
$$K_4^3 = \{(\{\omega_1, \omega_2\}, 0.3), (\{\omega_1, \omega_3, \omega_4\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$$

and

$K_1^4 = \{(\{\omega_1, \omega_2\}, 0.4), (\{\omega_2\}, 0.4)\}$
$K_2^4 = \{(\{\omega_1, \omega_2\}, 0.4), (\{\omega_1\}, 0.4), (\{\omega_1, \omega_4\}, 0.4)\}$
$K_3^4 = \{(\{\omega_1, \omega_3\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_3\}, 0.4)\}$
$K_4^4 = \{(\{\omega_2, \omega_4\}, 0.4), (\{\omega_4\}, 0.4), (\{\omega_1, \omega_4\}, 0.5)\}$

TABLE III
POSSIBILITY DISTRIBUTIONS FOR THE 3RD AND 4TH SETS OF PIBS

set	PIB	π	ω_1	ω_2	ω_3	ω_4	ω_5
3	K_1^3	π_1^3	0.5	0.6	1.0	0.6	0.5
	K_2^3	π_2^3	1.0	0.6	0.6	0.5	0.5
	K_3^3	π_3^3	0.6	1.0	0.6	0.6	0.6
	K_4^3	π_4^3	1.0	0.5	0.6	0.7	0.5
4	K_1^4	π_1^4	0.6	1.0	0.6	0.6	0.6
	K_2^4	π_2^4	1.0	0.6	0.6	0.6	0.6
	K_3^4	π_3^4	0.5	0.6	1.0	0.6	0.5
	K_4^4	π_4^4	0.6	0.5	0.5	1.0	0.5

If we set $\epsilon_0 = \epsilon_1 = 0.5, \epsilon_2 = 0.8$, based on Definition 7, the PIBs in the 3rd set can be merged with relaxation, since the final coherence interval is $[0.702, 1.0]$, the degree of inconsistency is 0.5 and the degree of information loss is 0.875. The PIBs in the 4th set can only be merged with the disjunctive rule, since conjunctively merging any pair of the PIB in the set will produce a rather lower coherence interval.

Now, let us re-examine the two sets of PIBs in Examples 2 and 3. The degree of information loss for both sets is 1.0 and the coherence intervals of the conjunctive merges of two sets are $I_{K_1} = [0.2439, 1.0]$ and $I_{K_2} = [0.4375, 1.0]$ respectively. Therefore, these two sets can only be merged with the disjunctive rule too, although the information loss is significant. However, the situation for these two sets is different from that for the 4th set. In fact, there is at least one subset in both the 1st and 2nd sets of PIBs that can be merged with relaxation. For the 1st set of PIBs in Example 2 this subset is $\{K_1^1, K_2^1, K_4^1\}$, and for the 2nd set of PIBs in Example 3 this subset can either be $\{K_1^2, K_3^2\}$ or $\{K_2^2, K_4^2\}$. Furthermore, $\mathsf{Inc}(K_{13}^2) = 0.0$. These simple facts have totally been overlooked when using a single merging rule and consequently, neither of the two rules alone is adequate for situations where there are some PIBs which are largely partially consistent and that can be merged with relaxation. $\diamond$

It should be pointed out that when comparing the qualities of disjunctively and conjunctively merged results of the same set of PIBs, we cannot apply the criteria stated in Definition 2 for individual PIBs, because the result of the disjunctive merge is always totally consistent. However, it does not imply that it is of better quality, due to the information loss during merging.

IV. GENERATING LARGELY PARTIALLY MAXIMAL CONSISTENT SUBSETS

To find out which subset of sources can be merged conjunctively (or with relaxation) in order to overcome the problem of applying only a single merging rule, such as faced by Example 5, we need to assess how similar the information from a pair of sources is. To do this, we define the following distance relation.

A. Distance between uncertain information

Definition 8 Let K_1, K_2 and K_3 be three PIBs. Let K_{12} and K_{13} be the conjunctively merged results of K_1 and K_2, and K_1 and K_3 respectively. A binary distance relation between K_2 and K_3 with reference K_1, denoted as $\preceq_{K_1}$, is defined as $K_2 \preceq_{K_1} K_3$ when one of the following conditions holds:
(a) π_{12} and π_{13} are normal and $H(\pi_{12}) \leq H(\pi_{13})$;
(b) $\mathsf{Inc}(K_{12}) \leq \mathsf{Inc}(K_{13})$;
(c) $\mathsf{Inc}(K_{12}) = \mathsf{Inc}(K_{13})$ and $I_{K_{13}} \leq_{lex} I_{K_{12}}$.
where π_{12} and π_{13} are the possibility distributions of K_{12} and K_{13}. When the qualities of K_{12} and K_{13} are indistinguishable (Definition 5), K_2 and K_3 are said to have the same distance with respect to K_1 and is denoted as $K_2 \sim_{K_1} K_3$.

$K_2 \preceq_{K_1} K_3$ indicates that the distance between K_1 and K_2 is not greater than that between K_1 and K_3, therefore, K_2 is more consistent with K_1 and should be merged with K_1 first before considering K_3. K_1 is called a *reference PIB*.

For instance, given the four PIBs in Example 2, the distance relation with reference K_1^1 is

$$K_2^1 \preceq_{K_1^1} K_4^1, \quad K_4^1 \preceq_{K_1^1} K_3^1,$$

since the coherence intervals of $K_{12}^1, K_{14}^1, K_{13}^1$ are $[0.636, 1.0], [0.50, 1.0]$ and $[0.238, 1.0]$ respectively, where K_{ij}^1 is the conjunctively merged result of K_i^1 and K_j^1. It suggests that K_2^1 is closer to K_1^1 than K_4^1 is, and K_4^1 is closer to K_1^1 than K_3^1 is.

Definition 9 Let $K, K_1,.., K_n$ be $n+1$ PIBs which have normal possibility distributions and K be the reference PIB. The *preferred sequence of merging* with reference K is defined as $(K, K_{i_1}, .., K_{i_j}, K_{i_{j+1}}, .., K_{i_n})$ such that for any $1 \leq j < n$, $K_{i_j} \preceq_{K^{j-1}} K_{i_t}$ when $j < t \leq n$, where K^{j-1} is the conjunctively merged result of the first j PIBs in the sequence. When $j = 1, K^0 = K$.

In this way, given a reference PIB, i.e., a source to start with, it is possible to order remaining PIBs according to their distances with the merged result of the already selected PIBs.

B. Generating LPMCSs

Definition 10 Let $(K, K_{i_1}, .., K_{i_j}, K_{i_{j+1}}, ..., K_{i_n})$ be the preferred sequence of merging with reference K as defined in Definition 9. Then $S_K = \{K, K_{i_1}, .., K_{i_j}\}$ is called the *largely partially maximal consistent subset (LPMCSs)* with reference K if the PIBs in S_K can be merged with relaxation, but PIBs in $S_K \cup \{K_{i_{j+1}}\}$ cannot.

This definition selects the first j PIBs that can be merged with relaxation. The merged result is within the inconsistency tolerance degree, has a coherence interval with the lower bound above the defined threshold. Furthermore, if these PIBs are merged disjunctively, the degree of information loss is above the acceptable level.

The choice of a reference PIB can be made based on the quality assessment of individual PIBs defined in Definition 5. Usually, we start with a PIB that has a better quality than other

PIBs, if there are several candidates, we choose the one that is provided by the most reliable source [12].

C. Effect of reference PIBs

In this subsection we investigate how much influence a chosen reference PIB has on a LPMCS generated by Definition 10. First we examine the four PIBs in Example 2. Table IV below summarizes the coherence interval after conjunctively merging the reference PIB with another PIB.

TABLE IV
COHERENCE INTERVALS OF MERGING A REFERENCE PIB WITH OTHER PIBS

Ref PIB	Coherence with	Coherence with	Coherence with
K_1	K_2, $[0.636, 1.0]$	K_3, $[0.238, 1.0]$	K_4, $[0.500, 1.0]$
K_2	K_1, $[0.636, 1.0]$	K_3, $[0.538, 1.0]$	K_4, $[1.0, 1.0]$
K_3	K_1, $[0.238, 1.0]$	K_2, $[0.538, 1.0]$	K_4, $[0.384, 1.0]$
K_4	K_1, $[0.500, 1.0]$	K_2, $[1.0, 1.0]$	K_3, $[0.384, 1.0]$

With information in this table, it is possible to initialize the preferred sequence of merging for each reference PIB as detailed in Column 2 in Table V using the distance relation defined in Definition 8. Then Definition 9 is applied to re-arrange the PIBs starting from the 3rd PIB in each initial sequence, and to generate the final preferred sequence of merging for each reference PIB. The final preferred sequence for each PIB is listed in Table VI.

TABLE V
THE INITIAL PREFERRED SEQUENCE FOR EACH REFERENCE PIB, AND THE COHERENCE INTERVALS AT THE 2ND ROUND. "2ND ROUND COHERENCE OF (1,2,3) (RESP., (1,2,4)) PIBS" MEANS THE COHERENCE INTERVAL AFTER MERGING THE 1ST, THE 2ND AND THE 3RD (RESP., (1ST, 2ND, 4TH)) PIBS CONJUNCTIVELY IN THE INITIAL SEQUENCE.

Ref PIB	Initial preferred sequence	2nd round coherence of (1,2,3) PIBs	2nd round coherence of (1,2,4) PIBs
K_1	K_1, K_2, K_4, K_3	$[0.580, 1.0]$	$[0.333, 1.0]$
K_2	K_2, K_4, K_1, K_3	$[0.580, 1.0]$	$[0.405, 1.0]$
K_3	K_3, K_2, K_4, K_1	$[0.405, 1.0]$	$[0.333, 1.0]$
K_4	K_4, K_2, K_1, K_3	$[0.580, 1.0]$	$[0.405, 1.0]$

TABLE VI
THE FINAL PREFERRED SEQUENCE AND THE MAXIMAL CONSISTENT SUBSETS WITH REGARD TO EACH REFERENCE PIB.

Ref PIB	Final preferred sequence	LPMCS with Ref PIB	Remaining PIBs
K_1	K_1, K_2, K_4, K_3	$\{K_1, K_2, K_4\}$	K_3
K_2	K_2, K_4, K_1, K_3	$\{K_2, K_4, K_1\}$	K_3
K_3	K_3, K_2, K_4, K_1	$\{K_3\}$	K_1, K_2, K_3
K_4	K_4, K_2, K_1, K_3	$\{K_4, K_2, K_1\}$	K_3

Let the thresholds for applying the relaxation of the rule in Definition 7 be set as $\epsilon_0 = \epsilon_1 = 0.5$ (ϵ_1 is the lower bound of the coherence interval) and $\epsilon_2 = 0.8$ (the degree of information loss). Then the LPMCS for reference K_1 is $\{K_1, K_2, K_4\}$, since this subset has a degree of inconsistency 0.4, a conjunctively merged coherence interval $[0.580, 1.0]$, and the degree of information loss in the disjunctive merge 1.0. If we chose K_2 or K_4 as the reference PIB, the LPMCSs are $\{K_2, K_4, K_1\}$ and $\{K_4, K_2, K_1\}$ respectively, which are all the same as that for K_1. Therefore, theoretically, any of the three can be chosen as a reference PIB to generate this LPMCS. For reference K_3, if we consider $\{K_3, K_2\}$ as a LPMCS, the degree of information loss when disjunctively merging these two PIBs is greater than the threshold, also, the lower bound of the coherence interval of merging K_2 and K_3 is below the required threshold (0.5), therefore, the only LPMCS for reference K_3 is K_3.

For this example, no matter which reference PIB we start with to generate potential LPMCSs, the only two such subsets are $\{K_1, K_2, K_4\}$ and $\{K_3\}$. Especially for subset $\{K_1, K_2, K_4\}$, choosing either K_1, or K_2 or K_4 has no effect on the final subset that contains them. However, whether this specific case is always true remains to be further investigated.

V. A CONTEXT-DEPENDENT ADAPTIVE MERGING ALGORITHM

A. The algorithm

Given a set of PIBs, there can be multiple LPMCSs that can be merged with the relaxation of the conjunctive rule. The following algorithm implements multiple subsets merging under these circumstances.

Algorithm Merging(λ): $\lambda = \{K_1, ..., K_n\}$ is a set of sources.

Begin
$m = 1$;
while $|\lambda| > 0$ **do**
  Let K_i be the PIB with the best quality in λ
    based on Def 5; (choose the one provided by the most
    reliable source if there is more than one candidate PIB)
  Let $S_m = \{K_i, K_{i_1}, .., K_{i_j}\}$ be the LPMCS generated
    based on Definition 10 with reference K_i,
    then calculate $\pi^m(\omega) = \min\{\pi_i(\omega), \pi_{i_1}(\omega), .., \pi_{i_j}(\omega)\}$;
  Let $\lambda = \lambda \setminus S_m$, $m = m + 1$;
End of while
$\pi^n(\omega) = \max\{\pi^1(\omega), .., \pi^{m-1}(\omega)\}$.
End

π^n is the final result of merging all the PIBs in λ. Let m be from Merging(λ), when $m = 2$, $\pi^n(\omega) = \min\{\pi_1(\omega), .., \pi_n(\omega)\}$, all the sources in λ are largely partially maximal consistent because they can all be merged with a single conjunctive rule (possibly with relaxation). When $m = n + 1$, $\pi^n(\omega) = \max\{\pi_1(\omega), .., \pi_n(\omega)\}$, all the sources are pairwise highly inconsistent, as they violate at least one of the conditions of applying the relaxation of the conjunctive rule.

In the following, we use π_{AL} to denote the final outcome of our algorithm.

It should be noted that the algorithm is nondeterministic if we have several choices in the first step, that is, the selection of reference PKB in the first step can influence subsequent PLMCSs. When several PIBs have the same best quality here, we need to make use of the knowledge of reliability of each source [12]. In this case, a source with a higher degree of reliability should be selected. Therefore, given a set of PIBs and necessary knowledge about the reliability of these sources, this merging algorithm has a unique output.

B. Computational complexity of the algorithm

Proposition 4 When applying the above algorithm to merge n PIBs, the computational complexity for performing conjunctive merges is $O(n^2)$ and the computational complexity for performing the calculation of coherence intervals and non-specificity together is $O(n^2)$ where $n = |\lambda|$.

Proof

According to Definition 10, the LPMCS with reference K_i is in fact the first j elements in the preferred sequence of merging for reference K_i such that these j PIBs can be merged with relaxation, but the first $j+1$ PIBs cannot.

We start with the full set of PIBs. Let the first chosen reference PIB be denoted as K_1 and the preferred sequence of merging for K_1 be (K_1) initially.

To decide the 2nd PIB in the preferred sequence of merging with K_1, it takes $n-1$ times of conjunctive merge with each merge combining K_1 with one of the remaining $n-1$ PIBs respectively. Assume the closest PIB to K_1 is K_2 without losing generality (Definition 8), the preferred sequence thus is revised as (K_1, K_2).

Similarly, it requires $n-2$ times of conjunctive merge to decide the 3rd PIB in the preferred sequence of merging with regard to the result of merging K_1 and K_2 (Definition 8). Let this PIB be named as K_3. This process is repeated until the $(i+1)^{th}$ PIB has been decided in the sequence which required $n-i$ times of merge. The latest preferred sequence is $(K_1, K_2, ..., K_{i-1}, K_i, K_{i+1})$.

Assume that the first i PIBs in the sequence can be merged with relaxation, but these $i+1$ PIBs together cannot, then the creation of the preferred sequence for K_1 stops here, because there is no need to order the remaining PIBs into the preferred sequence any further. Therefore, the first LPMCS is created with i PIBs. The total number of conjunctive merges performed so far is

$$N1 = (n-1) + (n-2) + ... + (n-(i-1)) + (n-i)$$

The total number of calculations of coherence intervals and degrees of non-specificity are the same as above (because we either calculate the coherence interval if the π concerned is not normal, or calculate the degree of non-specificity if the π concerned is normal, but not both).

Analogous to the above steps in creating the first LPMCS, we can create the next LPMCS after deleting these i PIBs in the initial set. Let the number of remaining PIBs be $n' = n-i$ and let the chosen reference PIB be K'. When the preferred sequence of merging with reference K' has been created with $j+1$ PIBs such that the first j PIBs can be merged with relaxation, but all these $j+1$ PIBs together cannot, then the second LPMCS is generated with j PIBs. The total number of conjunctive merges performed in this around is

$$\begin{aligned} N2 &= (n'-1) + (n'-2) + ... + (n'-(j-1)) + (n'-j) \\ &= (n-i-1) + (n-i-2) + ... + (n-i-j) \end{aligned}$$

Therefore, after the first two LPMCSs are created with $i+j$ PIBs, the total number of conjunctive merges performed is the sum of $N1$ and $N2$ which is

$$N3 = (n-1) + (n-2) + ... + (n-(i+j))$$

In a similar fashion, it is possible to deal with the remaining $n-(i+j)$ PIBs until all the PIBs are in a LPMCS (including LPMCSs having only one PIB) and the total number of conjunctive merges performed in all these rounds is

$$\begin{aligned} N &= (n-1) + (n-2) + ... + (n-l) + ... + (n-(n+2)) \\ &= \tfrac{(n-1)n}{2} + 1 \end{aligned}$$

so is the total number of calculations of coherence intervals and degrees of non-specificity. Therefore the computational complexity for both is $O(n^2)$. $\diamond$

C. Merging with the algorithm: an example

Example 6 We now apply the algorithm to the four sets of PIBs given in Examples 2, 3, and 5 to see the significance of the algorithm.

Applying $\mathsf{Merging}(\lambda)$ to $\lambda = \{K_1^1, K_2^1, K_3^1, K_4^1\}$ produces a subset $\{K_1^1, K_2^1, K_4^1\}$ which is conjunctively mergable with relaxation, because the coherence interval of the subset is $[0.580, 1]$, the inconsistency degree is 0.4, and $\mathsf{InfoLoss}(K_{124}^1) = 1.0$, where K_{124}^1 is the disjunctive merge result. The final possibility distribution from the algorithm is

$$\pi_{AL}^1(\omega) = \max\{\min(\pi_1^1(\omega), \pi_2^1(\omega), \pi_4^1(\omega)), \pi_3^1(\omega)\}$$

which is normal. This result retains more information than either the simple conjunctive or disjunctive merge, although the last step is a disjunctive merge.

Similarly, applying the algorithm to the other three sets of PIBs, we have

$$\begin{aligned} \pi_{AL}^2(\omega) &= \max\{\min(\pi_1^2(\omega), \pi_3^2(\omega)), \min(\pi_2^2(\omega), \pi_4^2(\omega))\} \\ \pi_{AL}^3(\omega) &= \min\{\pi_1^3(\omega), \pi_2^3(\omega), \pi_3^3(\omega), \pi_4^3(\omega)\} \\ \pi_{AL}^4(\omega) &= \max\{\pi_1^4(\omega), \pi_2^4(\omega), \pi_3^4(\omega), \pi_4^4(\omega)\} \end{aligned}$$

$\diamond$

When the only merging method applicable is disjunctive and the degree of information loss becomes too significant in the last step, as for the 4th set of PIBS, some sources may have to be discarded. The issue on how to reject certain sources will be one of the future research topics.

VI. Comparison with related work

There have been several adaptive merging rules that either revise the conjunctive operator with the degree of inconsistency or integrate both conjunctive and disjunctive operators when merging conflict sources of information. In this section, we compare the final outcome of our algorithm with some of the context dependent merging approaches reported in the literature.

A. Adaptive merging rule by Dubois and Prade

Dubois and Prade's adaptive rule. In [11], an adaptive rule was proposed which integrates both conjunctive and disjunctive merges

$$\pi_{DP}(\omega) = \max\left\{\frac{\pi_{cm}(\omega)}{h_{cm}}, \min(1 - h_{cm}, \pi_{dm}(\omega)\right\} \qquad (8)$$

where $\pi_i, i = 1, .., n$ are the n original possibility distributions, π_{cm} and π_{dm} are the conjunctively and disjunctively merged possibility distributions of these n sources, and $h_{cm} = h(\pi_1, .., \pi_n) = \max_{\omega\in\Omega}\{\min\{\pi_1(\omega_i), .., \pi_n(\omega_i)\}\}$ obtains the maximum degree of possibility of the conjunctive merge. In fact, $h_{cm} = 1 - \mathrm{Inc}(K_{cm})$.

special cases: When $h(\pi_1, .., \pi_n) = 1.0$, these sources are totally consistent (see Definition 2). π_{DP} is reduced to π_{cm}, so is the outcome of our algorithm. Therefore, $\pi_{DP} = \pi_{AL}$, that is, Dubois and Prade's adaptive rule and our algorithm are equivalent.

When $h(\pi_1, .., \pi_n) = 0.0$, these sources are totally inconsistent (see Definition 2). Equation 8 is reduced to $\pi_{DP} = \pi_{dm}$. However, π_{AL} is not necessarily reduced to π_{dm} since some of the original PIBs may still be mergable with relaxation. Therefore, $\pi_{AL}(\omega_i) \leq \pi_{dm}(\omega_i)$ in general. That is, our algorithm gives a more specific possibility distribution after merging than that from Equation 8.

General cases: To examine the outcome of both the Rule 8 and our algorithm for randomly created PIBs, we look at the four sets of PIBs in Examples 2, 3, and 5 again.

The conjunctively merged possibility distributions for the first three sets are the same and the details of the degrees of possibility distributions are

$\pi_{cm}(\omega_1) = 0.5; \pi_{cm}(\omega_2) = 0.5; \pi_{cm}(\omega_3) = 0.6$
$\pi_{cm}(\omega_4) = 0.5; \pi_{cm}(\omega_5) = 0.5;$

The fourth set has the conjunctive possibility distribution as

$\pi_{cm}(\omega_1) = 0.5; \pi_{cm}(\omega_2) = 0.5; \pi_{cm}(\omega_3) = 0.5$
$\pi_{cm}(\omega_4) = 0.6; \pi_{cm}(\omega_5) = 0.5;$

As a consequence, the degrees of consistency of the merged possibility distributions are the same too, which is $h_{cm} = 0.6$. The disjunctively merged possibility distributions of the four sets of PIBS are slightly different with details as shown in Table VII.

TABLE VII
THE DISJUNCTIVELY MERGED POSSIBILITY DISTRIBUTIONS FOR THE FOUR SETS OF PIBS IN EXAMPLES 2, 3, AND 5.

set	Merged π	ω_1	ω_2	ω_3	ω_4	ω_5
1	π^1_{dm}	1.0	1.0	1.0	1.0	0.5
2	π^2_{dm}	1.0	1.0	1.0	0.7	0.5
3	π^3_{dm}	1.0	1.0	1.0	0.7	0.6
4	π^4_{dm}	1.0	1.0	1.0	1.0	0.6

The adaptive rule in Equation 8 is then applied to these four sets to obtain the four merged results. These results are compared with the outcome obtained by our algorithm to the same four sets of PIBs. Table VIII below summarizes these merged possibility distributions.

TABLE VIII
THE MERGED RESULTS FROM DUBOIS AND PRADE'S RULE AND OUR ALGORITHM FOR THE FOUR SETS OF PIBS. THE SUPERSCRIPT ON π IN THE 2ND COLUMN REFERS TO WHICH SET OF PIBS IS BEING MERGED.

Set	Merged π	ω_1	ω_2	ω_3	ω_4	ω_5
1	π^1_{DP}	0.833	0.833	1.0	0.833	0.833
1	π^1_{AL}	0.5	0.6	1.0	0.6	0.5
2	π^2_{DP}	0.833	0.833	1.0	0.833	0.833
2	π^2_{AL}	1.0	1.0	0.6	0.5	0.5
3	π^3_{DP}	0.833	0.833	1.0	0.833	0.833
3	π^3_{AL}	0.5	0.5	0.6	0.5	0.5
4	π^4_{DP}	0.833	0.833	0.833	1.0	0.833
4	π^4_{AL}	1.0	1.0	1.0	1.0	0.6

Analysis: First of all, the results from applying Eq (8) on all the four sets of sources are almost identical. In fact, they have the same degree of nonspecificity. With the merged results alone, one cannot tell that some of these sets contain subsets within them that are largely consistent and can be merged with relaxation. Therefore, although this adaptive rule integrates both the conjunctive and disjunctive operators and the degree of consistency, it is not adaptive enough to deal with subsets with a set of sources and therefore cannot reflect the differences among different sets of sources.

Second, the results from our algorithm change from set to set, clearly indicating that these sets of sources are different. Since we require that all the original sources are normal, an unnormalized merged possibility distribution, such as π^3_{AL}, concludes that an unnormalized merged π can only be the result of merging all the sources conjunctively. Therefore, these sources are merged with relaxation and they must be largely partially consistent. On the other hand, if there are many elements having the maximum degree of possibility (e.g., 1) after merging, for instance, π^4_{AL}, it strongly hints that this can be the result of disjunctive merge. Therefore, these original sources are pair-wise largely inconsistent (otherwise, a subset may have been merged with relaxation) and one should treat the merged result with caution. This warning message cannot be reflected by Eq (8).

Although both the 1st and the 2nd merged possibility distributions are normal, there are only a small number of elements with the maximum degree of possibility, so they can either be the result of conjunctively merging consistent sources or of performing a disjunctive merge in the final step. The algorithm is able to tell precisely which operator has been used in the last step.

Third, for a set of sources which are largely partially consistent, the merged result from the algorithm is more specific than that from Eq (8) (this can also be seen from comparing $H(\pi_{AL})$ with $H(\pi_{DP})$), so the former provides a better final outcome.

One last note about the rule in Eq (8). As pointed out by

Dubois and Prade that the rule given in Eq (8) is more suitable for merging two sources, since it only considers two assumptions "all sources are right" (corresponding to conjunctive) and "one source is right" (corresponding to disjunctive). In other words, it lacks context dependency when merging more than two sources. Therefore, overall, we believe that our algorithm offers a more context dependent approach to merging multiple sources.

B. Subsets merging by Dubois and Prade

The main spirit behind our algorithm is to find LPMCSs that can be merged with relaxation to reinforce the beliefs among the sources in each LPMCS. The criteria to find these LPMCSs is based on the coherence intervals of the merged results of the PIBs in a subset, the degree of information loss if merged disjunctively, and the degree of inconsistency. In the case of our algorithm, these subsets are generated dynamically with respect to what remains to be merged. Because of this, the number of LPMCSs generated is determined solely by the information provided.

This scenario was observed and discussed in [14]. Unlike our argument which takes a number of factors into account when deciding LPMCSs, the scenario in [14] assumes that "a subset $J \subseteq K$ of experts such that $|J| = j$ are assumed reliable and their opinions (should be merged) conjunctively". Considering that it is not known which j experts are reliable, all the subsets with cardinality j are considered. The intermediate conjunctively merged results are then merged disjunctively. Mathematically this scenario is described as

$$\pi_{(j)}(\omega) = \max_{J \subseteq K, |J|=j}\{\min_{i \in J}(\pi_i(\omega) | \omega \in \Omega)\}$$

Obviously, the difficulty with this expression is to decide the number of reliability experts j. One way to do this is to apply the following condition [12]. For a subset of experts (sources) T, let

$$m = \max\{|T| \mid h(T) = 1\}, n = \max\{|T| \mid h(T) > 0\}$$

where $h(T) = h_{cm}$, the degree of consistency after merging information from the sources conjunctively.

These conditions first find the cardinality of the largest subset in which the sources can be merged conjunctively (value m) and then find the maximum number of sources (value n) that after merging these sources, the degree of consistency is greater than 0.0. Once the cardinality of the subset is decided, all the subsets with the same cardinality are merged conjunctively respectively before merging their results disjunctively. This method, therefore, can select the largest subset in which the PIBs are totally consistent. It is with this sense that it is one step closer to context-dependent merging than the rule in Eq (8).

There are two main drawbacks with this scenario. First, enumerating all the subsets with the same size is computationally expensive and practically unnecessary, since the subset which can generate the consistent result is known already. Second, if there are multiple disjoint subsets with different cardinalities, each of them has a set of PIBs that can be merged with relaxation, then the subsets with lower cardinalities are all drowned by the largest subset, since this method only consider those subsets having the same cardinalities with the largest one. Therefore, it is not possible to deal with consistent subsets with different sizes independently.

The major advantage of our algorithm over this subset merging is that we have multiple LPMCSs for conjunctive merges (possibly with relaxation). This can be reflected with the four sets of PIBs used in the previous section. The comparison of merging them using Dubois and Prade's subset method and our algorithm is summarized in Table IX. It is clear that Dubois and Prade's method is reduced to the disjunctive merge situation most of the time unless there exists at least one subset that can be merged conjunctively. Overall, their method is less informative than the adaptive rule given in Eq 8, and both are less informative then the outcome of our algorithm.

TABLE IX
THE MERGED RESULTS FROM DUBOIS AND PRADE'S SUBSET MERGING AND OUR ALGORITHM FOR THE FOUR SETS OF PIBS. THE SUPERSCRIPT ON π IN THE 2ND COLUMN REFERS TO THE SET OF PIBS BEING MERGED.

Set	Merged π	ω_1	ω_2	ω_3	ω_4	ω_5
1	π_j^1	1.0	1.0	1.0	1.0	0.5
1	π_{AL}^1	0.5	0.6	1.0	0.6	0.5
2	π_j^2	1.0	1.0	0.6	0.6	0.5
2	π_{AL}^2	1.0	1.0	0.6	0.5	0.5
3	π_j^3	1.0	1.0	1.0	0.7	0.6
3	π_{AL}^3	0.5	0.5	0.6	0.5	0.5
4	π_j^4	1.0	1.0	1.0	1.0	0.6
4	π_{AL}^4	1.0	1.0	1.0	1.0	0.6

C. Other adaptive rules

There are a few more adaptive rules which are mainly evolved from revising the conjunctive merge, in contrast to the mixture of both conjunctive and disjunctive merges discussed above. These rules are

$$\pi_{A1}(\omega) = \max\left\{\frac{\pi_{cm}(\omega)}{h_{cm}}, 1 - h_{cm}\right\} \quad (9)$$

$$\pi_{A2}(\omega) = \min\left\{1, \frac{\pi_{cm}(\omega)}{h_{cm}} + 1 - h_{cm}\right\} \quad (10)$$

$$\pi_{A3}(\omega) = \pi_{cm}(\omega) + 1 - h_{cm} \quad (11)$$

Eq (11) is also known as one of the three commonly used normalization rules when a conjunctive merge produces an unnormalized π.

For the two extreme situations where $h(\pi_1, .., \pi_n) = 1.0$, all the three rules are equivalent to the conjunctive merge whilst when $h(\pi_1, .., \pi_n) = 0.0$, all the rules assign the degree of possibility 1.0 to every element in the frame.

For other cases, we have detailed the analysis in Table X with the four sets of PIBs used in previous sections. As we can see, rule in Eq (9) behaves in a very similar fashion to Eq (8), as long as $1 - h < \pi_{dm}(\phi)$ holds for most of $\phi \in \Omega$. It is also clear that none of these rules is better than either Eq (8) or our algorithm. Therefore, these rules are less adaptive and useful in a highly conflicting situation, a situation where most of the pairs of PIBs are highly inconsistent.

TABLE X
THE MERGED RESULTS FROM THE THREE ADDITIONAL RULES FOR THE FOUR SETS OF PIBS IN EXAMPLES 2, 3, AND 5.

Set	Merged π	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
1	π^1_{A1}	0.833	0.833	1.0	0.833	0.833
1	π^1_{A2}	1.0	1.0	1.0	1.0	1.0
1	π^1_{A3}	0.9	0.9	1.0	0.9	0.9
2	π^2_{A1}	0.833	0.833	1.0	0.833	0.833
2	π^2_{A2}	1.0	1.0	1.0	1.0	1.0
2	π^2_{A3}	0.9	0.9	1.0	0.9	0.9
3	π^3_{A1}	0.833	0.833	1.0	0.833	0.833
3	π^3_{A2}	1.0	1.0	1.0	1.0	1.0
3	π^3_{A3}	0.9	0.9	1.0	0.9	0.9
4	π^4_{A1}	0.833	0.833	0.833	1.0	0.833
4	π^4_{A2}	1.0	1.0	1.0	1.0	1.0
4	π^4_{A3}	0.9	0.9	0.9	1.0	0.9

VII. INTEGRATING RELIABILITY INTO MERGING

Resolving conflict among information from multiple sources can also be partially achieved by integrating the reliability of each source into the merging process. By crediting some sources over others, it is hoped that a conflict situation can be resolved.

A. Methods of integrating reliability into merging

There have been many investigations into how reliability information can be used in the merging [1], [20], [15], [5]. A common assumption with these investigations is that reliability of information for each source is extra information that has to be provided separately.

A linear weighted approach irrelevant to the merging operator was originally reported in [6], and is then used to merge information from two sources in [5] as detailed below.

$$\begin{aligned}\pi_{Merge}(\omega) = P(w_1,w_2)\pi_{\otimes}(\omega) + P_1(w_1,w_2)\pi_1(\omega) \\ +P_2(w_1,w_2)\pi_2(\omega)\end{aligned}$$

where w_1 and w_2 are the weights associated with the two sources respectively, and coefficients $P(w_1,w_2)$, $P_1(w_1,w_2)$ and $P_2(w_1,w_2)$ are assigned to the merged possibility distribution, the two original distributions respectively, and $\pi_{\otimes}$ is the merged result when operator $\otimes$ is replaced with either a conjunctive or a disjunctive operator. These coefficients are constrained by the following conditions.

$$P_1(w_1,w_2) = \begin{cases} 0, & \text{when } w_1 \le w_2 \\ \frac{w_1-w_2}{w_1} & \text{otherwise} \end{cases}$$

$$P_2(w_1,w_2) = \begin{cases} 0, & \text{when } w_2 \le w_1 \\ \frac{w_2-w_1}{w_2} & \text{otherwise} \end{cases}$$

$$P(w_1,w_2) = \begin{cases} 1, & \text{when } w_1 = w_2 = 0 \\ \frac{min\{w_1,w_2\}}{max\{w_1,w_2\}} & \text{otherwise} \end{cases}$$

where $P(w_1,w_2)+P_1(w_1,w_2)+P_2(w_1,w_2) = 1$. With these coefficients, information from one source can be preferred over another, or both sources can be preferred equally. For example, when $w_1 = 0.6$ and $w_2 = 0.4$, $P(0.6,0.4) = 0.667, P_1(0.6,0.4) = 0.333, P_2(0.6,0.4) = 0.0$, information from source one is particularly added into the final merged result.

Another common linear approach to adding weights to sources during merging is the convex sum of sources in the probability framework

$$\pi_{M1}(\omega) = \sum_i \lambda_i \pi_i(\omega) \text{ with } \sum_i \lambda_i = 1, \omega \in \Omega$$

where λ_i and π_i are the reliability and the possibility distribution of source i. This approach can deal with more than two sources, unlike the previous method, but does not integrate the fusion operator.

In [23], this rule is revised to integrate a merging operator, such as a disjunctive one as

$$\pi_{M2}(\omega) = \beta \sum_i \lambda_i \pi_i(\omega) + (1-\beta)\pi_{dm}(\omega)$$

with $\sum_i \lambda_i = 1$ and $\beta \in [0,1]$.

A more general fusion rule that cooperates both the reliability and merging operator was proposed in [1]

$$\begin{aligned}\pi_{M3}(\omega) = Xor(t_i)\mathsf{max}_{i=1}^n[t_i\pi_i(\omega)] \\ +And(t_i)\mathsf{min}_{i=1}^n[t_i\pi_i(\omega)]\end{aligned} \quad (12)$$

where operators Xor and And are the fuzzy equivalent of the logical operators.

In all the three rules listed above, the reliability values t_i or λ_i are explicitly required, which can be difficult to obtain for many applications, especially when there is no prior knowledge about the sources involved.

B. Modifying source information before merging

Instead of attaching a reliability to each source, another way of resolving a conflict among multiple sources is to revise some possibility distributions before merging. Two commonly used rules to revise possibility distributions are

$$\pi_Y(\omega) = t\pi(\omega) + 1 - t, \quad \pi_{DP'}(\omega) = \mathsf{max}\{\pi(\omega), 1-t\}$$

where t is interpreted as the degree of certainty that the source is reliable. The former was defined in [22] and the latter was proposed in [10].

When a source is completely unreliable, e.g., $t = 0.0$, the possibility distribution is reduced to a uniform one in both cases. When a source is complete reliable, the possibility distribution is unchanged.

Once again, the knowledge of t must be given in order to revise these distributions. Revising a possibility distribution in this way is also called discounting a possibility distribution.

VIII. Conclusion

In this paper, we proposed a context-dependent merging algorithm that dynamically generates largely partially maximal consistent subsets (LPMCSs) for conjunctive merges before a final disjunctive merge. To achieve this, we calculate the coherence interval after each step of conjunctive merge and the information loss of the corresponding disjunctive merge to assess if the relaxation of conjunctive merge is feasible. We also measure the distances among sources with a reference PIB and rank sources according to their information quality, so that the algorithm starts with a source of high quality. The degree of information loss of a disjunctive merge is measured to ensure that a disjunctive merge will not be carried out if it will cause too much loss of original information.

We have proved that the computational complexity of our algorithm is $O(n^2)$ with n original sources.

The comparison with related adaptive rules, especially with Dubois and Prade's rule and their subset merging scenario, shows that our algorithm is more adaptive to the context and can provide a more rational merge result.

The idea of finding the maximal consistent subsets among a given set of sources was presented in [7], [13]. These methods assume that the original information in each source is provided in the form "$x \in E_i \subseteq U$" which means source i believes that variable x is in subset E_i of the universe. When all the original information is provided in this way, that is, E_i is taken as an interval with $E_i = [a_1, b_i]$, then the boundaries of all the subsets can be arranged along a real number line. An efficient algorithm is thus designed to find all the maximal consistent subsets. The maximal consistent subset of sources with interaction $[a_i, b_j]$ is defined by the maximal number of elements of $\{E_1, ..., E_n\}$ which contain $[a_i, b_j]$. Obviously, a source with information $x \in E_i \subseteq U$ implies that $\pi_i(\omega_i) = 1$ for $\omega \in E_i$. Information in this form is a special case of information that we have considered in the paper and it can equally be dealt with by our algorithm. Our algorithm finds the same set of maximal consistent subsets when information is in this form and when it is required that each LPMCS is actually a maximal consistent subset, e.g., each LPMCS can be merged with the conjunctively rule directly.

In most research, e.g., [1], reliability is required as extra data in order to prioritize sources and, often, this extra information is not available. Given that coherence intervals can measure the quality of a source to some extend and that the degree of information loss can tell how suitable a disjunctive merge is, our next step of research will focus on how to provide more accurate information on reliability of a source from the information itself and how to use both the degree of information loss and coherence intervals to further guide the fusion process.

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