Instantiating Abstract Argumentation with Classical Logic Arguments: Postulates and Properties

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Abstract

In this paper we investigate the use of classical logic as a basis for instantiating abstract argumentation frameworks. In the first part, we propose desirable properties of attack relations in the form of postulates and classify several well-known attack relations from the literature with regards to the satisfaction of these postulates. Furthermore, we provide additional postulates that help us prove characterisation results for these attack relations. In the second part of the paper, we present postulates regarding the logical content of extensions of argument graphs that may be constructed with classical logic. We then conduct a comprehensive study of the status of these postulates in the context of the various combinations of attack relations and extension semantics.

Keywords: Computational models of argument, Abstract argumentation, Logical argumentation

1. Introduction

Argumentation is an important cognitive process that involves the generation and evaluation of arguments. There have been a number of proposals for capturing this cognitive process in computational models of argumentation (for example [1, 2, 3, 4, 5, 6] and for reviews [7, 8, 9, 10]). Amongst these proposals, two significant, intersecting and non-exclusive streams can be distinguished.

Abstract argumentation which focuses on the attack relations between arguments and usually considers arguments themselves to be atomic objects (for example, [11, 3, 12, 13]). This approach offers insight into how arguments interact and achieve acceptability solely in terms of the *attacks* that may exist between them. Furthermore, this approach allows for harnessing tools from graph theory.

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Logical argumentation which considers arguments as complex entities with an internal structure that is governed by a certain logical language (for example [1, 14, 4, 5, 6, 15]). In general, using logic for formalising argumentation enables the harnessing of natural concepts for disagreement, or attack between arguments, such as inconsistency. In addition, this approach allows using logical entailment for drawing conclusions that may serve as the claims of arguments.

A wealth of research has been conducted in the context of these two streams, prompting the question whether their derived knowledge can be combined to deliver models of argumentation that are more expressive, more natural or more powerful. Proposals of frameworks that are situated in the intersection of these two areas exist (for example, [16, 4, 5, 17, 18, 19, 20]) but (apart from [16, 19, 20]), they tend to focus on specialised, defeasible logics as their language of choice. Defeasible logics are a useful tool for many application areas where the expressiveness requirements are not great, but cannot readily encompass applications where for example disjunction or true negation are required.

Therefore, it would seem that an argumentation system that uses classical logic as its language, while explicitly employing abstract argumentation-style semantics, is attractive for several reasons. It would benefit directly from the large number of results on abstract argumentation frameworks produced in the last decade and, in addition, it would offer expressiveness that is moderate but still higher than currently used defeasible logics for argumentation.

It is the exploration of this gap, then, that this paper is aimed at. Proposing such a framework is not particularly complicated as there is a natural, if not unproblematic, way to instantiate an abstract argumentation framework using a knowledge base in classical logic. What is little known about such systems is the properties they ought to satisfy for them to be useful and predictable. While attempts have been made along these lines (e.g., [18] on rule-based systems and [19] on classical logic) such work is far from complete. Moreover, after such postulates have been proposed and perhaps gained acceptance, the more important question remains as to which argumentation systems will satisfy them. Compounding the difficulty of these questions is the fact that components of both logical and abstract argumentation, such as attack relations and extension semantics respectively, can be instantiated to a multiplicity of definitions, thus considerably enlarging the number of combinations to check. So we see that given the interesting possibilities raised in [16, 18, 19], there is a need to undertake a systematic analysis of instantiating abstract argumentation with classical logic that considers a comprehensive range of attack relations and semantics for extensions. Furthermore, it is desirable that this is undertaken in a modular way using a framework of intuitive postulates.

Given these concerns, in this paper we consider an intuitive way to generate a set of arguments, starting from a knowledge base in classical logic (along with preliminaries, in Section 2). We then review the attack relations for logical argumentation in the literature and set out postulates that delineate desirable properties (Section 3). In addition, we investigate the status of these properties for these attack relations and then go on to propose additional postulates that allow us to prove characterisation results, i.e., to show that an arbitrary attack relation is one of the reviewed ones if and only if it satisfies a particular combination of postulates. Following that, we outline a set of postulates which express general properties of the logical content of the extensions of logical argument graphs (Section 4). To achieve that, we employ and generalise properties found in the literature as well as novel postulates that specifically address the issues arising from the potential extension multiplicity that many extension semantics allow. The status of these postulates is examined next (Section 5), over the several combinations of extension semantics and attack relations possible. Finally, we conclude and discuss existing work that relates to this paper (Section 6).

2. Preliminaries

We will use a propositional logic $\langle \mathcal{L}, \vdash \rangle$ with a countable set of propositional letters and constants $\top, \bot$ for truth and falsum respectively. We write $\Phi \vdash \psi$ to mean that the set of formulae Φ entails the formula ψ, and $\phi \vdash \psi$ as shorthand for $\{\phi\} \vdash \psi$. The notation $\bigwedge \Phi$ where Φ is a set of formulae, will be used to denote the conjunction of all formulae in Φ. We also use $\phi \equiv \psi$ to denote logical equivalence of the formulae ϕ and ψ in the meta-language (i.e., $\phi \vdash \psi$ and $\psi \vdash \phi$), and $\Phi \equiv \Psi$ mean logical equivalence of sets of formulae, i.e., $\bigwedge \Phi \equiv \bigwedge \Psi$. From now on Δ, the knowledge base, will stand for a finite set of individually consistent propositional formulae. We will denote the set of minimal inconsistent subsets of Δ with $\mathsf{MI}(\Delta)$ (that is to say, $\mathsf{MI}(\Delta) = \{ \Phi \subseteq \Delta \mid \Phi \vdash \bot \text{ and for all } \Psi \subset \Phi, \Psi \nvdash \bot \}$).

In order to give meaning and structure to the arguments in an abstract argumentation framework, we adopt the most common definition in logical argumentation that separates the evidence, or support, from the claim, or conclusion of an argument.

Definition 1. *An* ***argument*** *is a pair $\langle \Phi, \phi \rangle$ such that $\Phi \subseteq \Delta$ is a consistent, finite set of formulae, ϕ is a formula such that $\Phi \vdash \phi$, and no proper subset of Φ entails ϕ.*

The (countably infinite) set of all arguments is denoted by $\mathcal{A}$. If $A = \langle \Phi, \phi \rangle$ is an argument, we will use the functions $\mathsf{S}(A) = \Phi$ to denote the ***support*** of A and $\mathsf{C}(A) = \phi$ to denote the ***claim*** of A. We say that two arguments A, B are equivalent if $\mathsf{S}(A) = \mathsf{S}(B)$ and $\mathsf{C}(A) \equiv \mathsf{C}(B)$, and denote this by $A \equiv B$. Notice that this notion of equivalence is semantic with respect to the claim, but syntactic with respect to the support. This compromise aims at accommodating the fact that users of a logical argumentation system may view the formulae in the input knowledge base as resources, in the sense that having two ways to prove the same thing should give rise to two different arguments that are not conflated together. Also, we will say that A is a ***sub-argument*** of B if $\mathsf{S}(A) \subseteq \mathsf{S}(B)$.

Definition 2. *An* ***argument graph*** *(or simply, graph) is a pair $\langle N, R \rangle$ where $N \subseteq \mathcal{A}$ is a finite set of arguments and R is a non-reflexive binary relation on N. For each $(A, B) \in R$, we say that A* ***attacks*** *B.*

We will only consider non-reflexive attack relations in this paper. While attack relations with self-loops have been of some interest in the abstract argumentation community, they do not normally feature in logical argumentation since arguments are usually required to be consistent, as above, and the attack relation definition ordinarily relates attack to inconsistency.

We use the convenience functions $\mathsf{Nodes}(\langle N, R \rangle) = N$ (the nodes of the graph) and $\mathsf{Arcs}(\langle N, R \rangle) = R$ (the arcs of the graph). Whenever we are concerned with an argument graph Γ and there is no possibility of confusion, we will understand the phrase 'an argument A in Γ' to mean the qualified phrase 'an argument $A \in \mathsf{Nodes}(\Gamma)$'.

We recall the following notions from abstract argumentation.

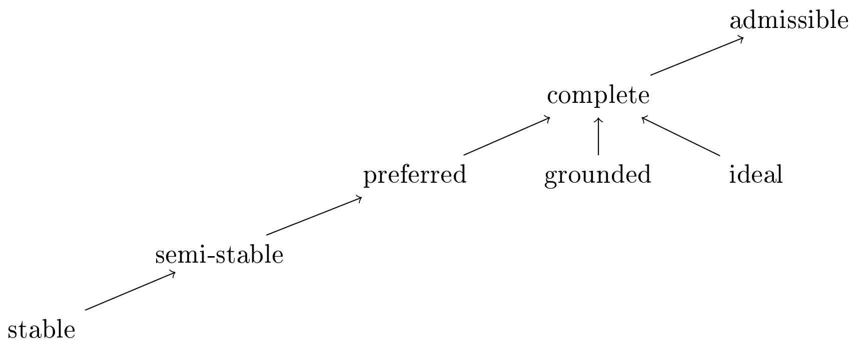


Figure 1: Is-a relationships between concepts in Definition 3. An arrow from X to Y denotes that any X-extension is also a Y-extension, or equivalently, $\mathcal{E}_{\mathrm{X}}(\Gamma) \subseteq \mathcal{E}_{\mathrm{Y}}(\Gamma)$.

Definition 3. *Let Γ be an argument graph and $S \subseteq \mathsf{Nodes}(\Gamma)$ a set of arguments.*

1. *S **attacks** an argument B if there exists $A \in S$ such that A attacks B.*
2. *S **defends** an argument A if S attacks B for every argument B such that B attacks A.*
3. *S is **conflict-free** if there is no $A \in S$ such that S attacks A.*
4. *S is **admissible** if it is conflict-free and it defends all of its members.*
5. *S is a **stable extension** if S is conflict-free and S attacks A for all $A \in \mathsf{Nodes}(\Gamma) \setminus S$.*
6. *S is a **complete extension** if it is admissible and contains every argument it defends.*
7. *S is a **preferred extension** if it is a maximal (w.r.t. set inclusion) admissible set.*
8. *S is the **grounded extension** if it is the least fixed point of $\mathcal{F}(X) = \{A \in \mathsf{Nodes}(\Gamma) \mid X$ defends $A\}$.*
9. *S is a **semi-stable extension** if it is a complete extension and the set $S \cup \{A \in \mathsf{Nodes}(\Gamma) \mid S$ attacks $A\}$ is maximal w.r.t. set inclusion.*
10. *S is the **ideal extension** if it is the maximal (w.r.t. set inclusion) admissible set that is contained within every preferred extension.*

Items 1–8 are due to Dung [3], item 9 is due to Caminada [12] and item 10 is due to Alferes, Dung and Pereira [11] and Dung, Mancarella and Toni [13]. The notions 5–10 are also referred to as argumentation semantics or extension semantics in the argumentation literature. Note that for a particular argument graph there may be no stable extensions. Also, there may be more than one stable, complete, preferred and semi-stable extensions, but only one grounded and one ideal extension. In addition, the is-a relationships shown in Figure 1 hold, proved in [3, 12, 13].

For any extension semantics X we will use $\mathcal{E}_{\mathrm{X}}(\Gamma)$ to denote the set of X-extensions of the argument graph Γ. We will use the subscripts: 'st' for stable, 'co' for complete, 'pr' for preferred, 'gr' for grounded, 'ss' for semi-stable and 'id' for ideal. So, for example, $\mathcal{E}_{\mathrm{pr}}(\Gamma)$ will stand for the set of preferred extensions of Γ. In addition, we define two sets of arguments for a graph Γ, meant to capture sceptical and credulous acceptance of arguments, respectively:

$$\pi_{\mathrm{X}}(\Gamma) = \bigcap_{S \in \mathcal{E}_{\mathrm{X}}(\Gamma)} S \qquad\qquad \sigma_{\mathrm{X}}(\Gamma) = \bigcup_{S \in \mathcal{E}_{\mathrm{X}}(\Gamma)} S$$

So, for example, we will say that an argument A in Γ is sceptically accepted in the preferred semantics if $A \in \pi_{\mathrm{pr}}(\Gamma)$. Clearly, $\pi_{\mathrm{X}}(\Gamma) \subseteq \sigma_{\mathrm{X}}(\Gamma)$ for any semantics X, $\sigma_{\mathrm{gr}}(\Gamma) = \pi_{\mathrm{gr}}(\Gamma)$ and $\sigma_{\mathrm{id}}(\Gamma) = \pi_{\mathrm{id}}(\Gamma)$. Note if $\mathcal{E}_{\mathrm{X}}(\Gamma) = \emptyset$, then we have $\pi_X(\Gamma) = \sigma_X(\Gamma) = \emptyset$.

Now that we have defined the notion of a logical argument, the question of how to define the notion of attack arises. In the context of abstract argumentation, an *attack relation* is, as we have seen, a binary relation on a set of arguments. While this is an exact description of what kind of mathematical object we use, it is of no help in explaining *how* we might define such a relation. Indeed, most, if not all, attack relations on logical arguments from the literature are defined by way of a criterion that operates on a pair of logical arguments and uses no other information about the set of arguments this pair belongs in. Thus, a natural way for expressing this is through a function. An ***attack function*** $D : \mathcal{A} \times \mathcal{A} \to \{\top, \bot\}$ is a boolean-valued function on ordered pairs of arguments. Such an attack function can be seen as the characteristic function of an attack relation, i.e., a set of pairs of arguments. For this reason from now on we will abuse terminology slightly and talk about attack functions and attack relations interchangeably.

We review below several attack relations from the literature.

Definition 4. *Let A and B be two arguments. We define the following attack functions by listing the conditions under which $D(A, B) = \top$. On the left we list the symbols for each attack function.*

(D_D) *A is a **defeater** of B if $\mathsf{C}(A) \vdash \neg \bigwedge \mathsf{S}(B)$.*

(D_{DD}) *A is a **direct defeater** of B if there is $\phi \in \mathsf{S}(B)$ such that $\mathsf{C}(A) \vdash \neg\phi$.*

(D_U) *A is an **undercut** of B if there is $\Psi \subseteq \mathsf{S}(B)$ such that $\mathsf{C}(A) \equiv \neg \bigwedge \Psi$.*

(D_{DU}) *A is a **direct undercut** of B if there is $\phi \in \mathsf{S}(B)$ such that $\mathsf{C}(A) \equiv \neg\phi$.*

(D_{CU}) *A is a **canonical undercut** of B if $\mathsf{C}(A) \equiv \neg \bigwedge \mathsf{S}(B)$.*

(D_R) *A is a **rebuttal** of B if $\mathsf{C}(A) \equiv \neg\, \mathsf{C}(B)$.*

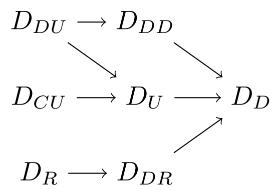
(D_{DR}) *A is a **defeating rebuttal** of B if $\mathsf{C}(A) \vdash \neg\, \mathsf{C}(B)$.*

It is straightforward to see that A is a defeater of B iff $\mathsf{S}(B) \cup \{\mathsf{C}(A)\}$ is inconsistent, and that A is a defeating rebuttal of B iff $\{\mathsf{C}(A), \mathsf{C}(B)\}$ is inconsistent.

The concepts behind these attack relations have been very widely employed in the literature, so citing originating papers with exactness is difficult. It is safe to say, however, that rebuttals appear in [1] and also [21]. Direct undercuts appear in [2, 22]. Undercuts and canonical undercuts were proposed in the above form and studied extensively in [23, 6]. Note that canonical undercuts were originally defined using the notion of *maximal conservativeness* but for simplicity we use the above equivalent definition.

Note that, if D_1, D_2 are two attack functions we will use them as relations in order to express a containment relation, e.g., $D_1 \subseteq D_2$, meaning that if $D_1(A, B) = \top$ then

Figure 2: Containment between attack relations. An arrow from D_1 to D_2 indicates that $D_1 \subseteq D_2$.



$D_2(A, B) = \top$. Then, it is easy to see that several containment relations hold as pictured in Figure 2.

Having chosen an attack function D and given a knowledge base Δ the question remains as to how an argument graph is generated. There is a natural definition for this, which we call an argument generator.

Definition 5. *The* ***argument generator*** $\mathbb{G}_D^\Delta$ *is defined as follows, given an attack function* D *and a knowledge base* Δ*.*

$$\begin{aligned}\mathbb{G}_D^\Delta &= \langle N, R\rangle\\ N &= \{\, A \in \mathcal{A} \mid \mathsf{S}(A) \subseteq \Delta \,\}\\ R &= \{\, (A, B) \mid A, B \in N \textit{ and } D(A, B) = \top \,\}\end{aligned}$$

By definition, it is always the case that for any formula ϕ and knowledge base Δ,

$$\mathsf{Nodes}(\mathbb{G}_D^\Delta) \subseteq \mathsf{Nodes}\left(\mathbb{G}_D^{\Delta\cup\{\phi\}}\right) \qquad \mathsf{Arcs}(\mathbb{G}_D^\Delta) \subseteq \mathsf{Arcs}\left(\mathbb{G}_D^{\Delta\cup\{\phi\}}\right).$$

In this paper, we have focused on the most commonly considered definition for a logical argument (i.e. $\langle \Phi, \psi\rangle$ is an argument iff Φ is a minimal consistent set of formulae that entails ψ). However, there are other proposals that relax these conditions for an argument. For instance, in assumption-based argumentation [17], there is not a requirement for the premises to be minimal. Further relaxations of these conditions for an argument have been considered in [24] resulting in approximate arguments such as enthymemes which have insufficient premises for entailing the claim.

An alternative definition for an argument has been proposed in [14], and revisited in [16], in which the consistency constraint is such that the premises Φ of an argument need to be consistent with the a designated subset of the knowledgebase. The idea behind this refinement is that there is a consistent subset Π of the knowledgebase that can be regarded as correct, and the support of any argument would need to be consistent with Π. This also leads to further instances of counterargument such as the following for arguments A and B:

- A disagrees with B iff $\Pi \cup \{\mathsf{C}(A), \mathsf{C}(B)\} \vdash \bot$.
- A counterargues B iff there is an argument C such that C is a subargument of B and A disagrees with C

- A weakly-undercuts B iff there is an argument C such that C is a subargument of B and A is a rebuttal of C

If we assume that $\Pi = \emptyset$, then we can compare these definitions with those given in Definition 4. For instance, A disagrees with B iff A is a defeating rebuttal of B. And, A counterargues B iff there is a subargument C of B and A is a defeating rebuttal of C.

3. Postulates concerning attack functions

In this section, we propose postulates relevant to attack functions. From now on, A, B, C and their primed versions will stand for arguments.

$(D0)$ $$\text{if } A \equiv A', B \equiv B' \text{ then } D(A,B) = D(A',B')$$

The above postulate is a classic syntax-independence requirement: the syntax of the components of two arguments should not play a role in deciding whether there is an attack between those arguments.

$(D1)$ $$\text{if } D(A,B) = \top \text{ then } \{\mathsf{C}(A)\} \cup \mathsf{S}(B) \vdash \bot$$

Postulate $D1$ mandates that if an argument attacks another, then it must be that the claim of the former is inconsistent with the support of the latter. This requirement reflects a fundamental assumption in logical argumentation, namely that if two arguments are logically consistent there cannot be any attack between them. In addition, a further expectation is that not only the supports of the arguments are inconsistent, but also the claim of the attacker with the support of the attacked argument. This fits in with the view that an argument can be decomposed into a claim and a set of *evidence* for that claim (here, the support). In this context, it is not enough to have inconsistent collections of evidence for an attack to take place; the attacking argument must make it specific in its claim that it contradicts the evidence offered by the attacked argument. Note also that since A is a defeater of B if and only if $\mathsf{S}(B) \cup \{\mathsf{C}(A)\}$ is inconsistent, we see that the $D1$ postulate mandates that an attack must be contained in the defeat relation.

$(D2)$ $$\text{if } D(A,B) = \top \text{ and } \mathsf{C}(C) \equiv \mathsf{C}(A) \text{ then } D(C,B) = \top$$

$(D3)$ $$\text{if } D(A,B) = \top \text{ and } \mathsf{S}(B) = \mathsf{S}(C) \text{ then } D(A,C) = \top$$

The postulates $D2$ and $D3$ impose certain fairness restrictions on existing attacks: $D2$ requires that all arguments that have equivalent *claims* with that of A should attack B; $D3$, similarly, requires that if A attacks B then all arguments with the same *support* with that of B should also be attacked by A. It is of note that these two postulates have some overlap with $D0$, but they are not equivalent, since in each case the new argument is not necessarily $\equiv$-equivalent to the existing argument.

It is possible to strengthen these postulates as follows. Note, $D3'$ was proposed by Amgoud and Besnard [19].

$(D2')$ $$\text{if } D(A,B) = \top \text{ and } \mathsf{C}(C) \vdash \mathsf{C}(A) \text{ then } D(C,B) = \top$$

$(D3')$ $$\text{if } D(A,B) = \top \text{ and } \mathsf{S}(B) \subseteq \mathsf{S}(C) \text{ then } D(A,C) = \top$$

Table 1: Postulates satisfied by the reviewed attack functions.

	D_D	D_{DD}	D_U	D_{DU}	D_{CU}	D_R	D_{DR}
$D0$	✓	✓	✓	✓	✓	✓	✓
$D1$	✓	✓	✓	✓	✓	✓	✓
$D2$	✓	✓	✓	✓	✓	✓	✓
$D2'$	✓	✓	×	×	×	×	✓
$D3$	✓	✓	✓	✓	✓	×	×
$D3'$	✓	✓	✓	✓	×	×	×
$D4$	✓	✓	✓	✓	✓	✓	✓

Again, suppose that A attacks B. Postulate $D2'$ requires that any argument with a stronger claim than A, i.e., one that logically entails that of A, should also attack anything A attacks. Postulate $D3'$ mandates that any argument whose support is a superset of that of B, and thus stronger than that of B, should also be attacked by A. Clearly, $D2'$ and $D3'$ entail their weaker versions, $D2$ and $D3$, respectively.

We list below another postulate that makes reference to the argument generator. This postulate essentially only constrains D in relation to the input knowledge base Δ.

$$(D4) \qquad \text{if } \mathsf{Arcs}(\mathbb{G}_D^{\Delta}) = \emptyset \text{ then } \mathsf{MI}(\Delta) = \emptyset$$

This postulate can be read as follows: if we restrict D on the arguments that can be generated from Δ and find that no two such arguments attack each other, then it must be that Δ itself is consistent (hence it has no minimal inconsistent subsets). It should be easy to see that $D1$ entails the right-to-left version of $D4$ and, therefore, these two postulates together entail that the resulting argument graph has no arcs iff Δ is consistent.

We examine now whether the attack functions described in Definition 4 satisfy the above postulates.

Proposition 6. *The attack functions in* $\{D_D, D_U, D_{CU}, D_{DU}, D_{DD}, D_R, D_{DR}\}$ *satisfy the attack function postulates according to Table 1.*

Proof. We will omit the proofs that are trivial but give some comments instead.

$D0$ is clearly satisfied by all the attack functions, as their definitions use logical equivalence or entailment for the claims and the subset relation or equality for the supports.

$D1$ is satisfied by all the attack functions.

$D2$ is satisfied by all the attack functions due to the fact that all attack function definitions put conditions on the claim of the attacking argument that cannot distinguish between logically equivalent claims.

$D2'$ will be satisfied by those attack functions that are tolerant to strengthening the claim of the attacking argument. Thus, all attack functions based on the notion of undercutting will fail this postulate.

$D3$ is satisfied by all the attack functions except for those based on the notion of rebuttal for obvious reasons.

$D3'$ is satisfied by all the attack functions that satisfy $D3$ except canonical undercutting, which by its definition does not tolerate changing the support of the attacked argument to a non-logically equivalent set.

$D4$ is satisfied by all attack functions. Suppose that $\mathsf{Arcs}(\mathbb{G}^{\Delta}_{D}) = \emptyset$ but $\mathsf{MI}(\Delta) \neq \emptyset$ and let $M \in \mathsf{MI}(\Delta)$ and $\psi \in M$. Since M is a minimal inconsistent set it is easy to see that $A = \langle M \setminus \{\psi\}, \neg\psi\rangle$ is an argument in $\mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$. Also, trivially, $B = \langle\{\psi\}, \psi\rangle$ is an argument in $\mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$. But for any attack function $D \in \{D_D, D_U, D_{CU}, D_{DU}, D_{DD}, D_R, D_{DR}\}$, $D(A, B) = \top$, thus $\mathsf{Arcs}(\mathbb{G}^{\Delta}_{D}) \neq \emptyset$, a contradiction.

□

The postulates $D0$-$D4$ are useful in classifying the attack functions we review, as evidenced by the table in Proposition 6. We can also use these postulates for a characterisation of the attack functions reviewed here. By characterisation we mean a result that guarantees that an arbitrary attack function D satisfies a specific list of postulates if and only if D is a specific member of $\{D_D, D_U, D_{CU}, D_{DU}, D_{DD}, D_R, D_{DR}\}$. To that end, we present a list of further postulates that will be useful in characterising the reviewed attack functions. We first look at postulate $D1$ and two variations.

$(D1')$ if $D(A, B) = \top$ then $\exists\phi \in \mathsf{S}(B)$ s.t. $\mathsf{C}(A) \vdash \neg\phi$

$(D1'')$ if $D(A, B) = \top$ then $\mathsf{C}(A) \vdash \neg\,\mathsf{C}(B)$

Where $D1$ requires inconsistency of the attacking arguments's claim and the attacked argument's support, $D1'$ and $D1''$ put more stringent requirements on the claim of the attacker. Also, note that $D1'$ and $D1''$ imply $D1$.

Next we introduce four postulates that work in tandem with one of the versions of $D1$ to provide one direction of the characterisation result. So each $D1$ variant works with a $D5$ variant to fix the nature of the relationship between the support of an argument and the claim of a counterargument. For instance, $D1''$ says that for $D(A, B) = \top$ to hold, $\mathsf{C}(A)$ entails $\neg\mathsf{C}(B)$, and $D5''$ says that for $D(A, B) = \top$ to hold, $\neg\mathsf{C}(A)$ entails $\mathsf{C}(B)$. Therefore, together $D1''$ and $D5''$ say that $\mathsf{C}(A)$ is equivalent to $\neg\mathsf{C}(B)$. It can be seen that $D5$ implies $D5'$ and $D5''$, and that $D5'''$ implies $D5'$.

$(D5)$ if $D(A, B) = \top$ then $\neg\,\mathsf{C}(A) \vdash \bigwedge \mathsf{S}(B)$

$(D5')$ if $D(A, B) = \top$ then $\exists\phi \in \mathsf{S}(B)$ s.t. $\neg\,\mathsf{C}(A) \vdash \phi$

$(D5'')$ if $D(A, B) = \top$ then $\neg\,\mathsf{C}(A) \vdash \mathsf{C}(B)$

$(D5''')$ if $D(A, B) = \top$ then $\exists X \subseteq \mathsf{S}(B)$ s.t. $\neg\,\mathsf{C}(A) \equiv \bigwedge X$

Finally, it is of note that all postulates presented up to now are universally quantified over arguments. This means that they cannot force the existence of attacks, but only constrain those present. Hence, the empty attack relation $D_{\emptyset}(A, B) = \bot$ will trivially satisfy all these postulates. Thus, for the purposes of providing characterisation results we need a set of postulates that guarantee the existence of attacks. We present these below.

$(D6)$	if $\{\mathsf{C}(A)\} \cup \mathsf{S}(B) \vdash \bot$ then there exists C s.t. $\mathsf{C}(A) \vdash \mathsf{C}(C)$ and $D(C, B) = \top$
$(D6')$	if $\exists \phi \in \mathsf{S}(B)$ s.t. $\mathsf{C}(A) \vdash \neg\phi$ then there exists C s.t. $\mathsf{C}(A) \vdash \mathsf{C}(C)$ and $D(C, B) = \top$
$(D6'')$	if $\mathsf{C}(A) \vdash \neg\, \mathsf{C}(B)$ then there exists C s.t. $\mathsf{C}(A) \vdash \mathsf{C}(C)$ and $D(C, B) = \top$
$(D6''')$	if $\exists X \subseteq \mathsf{S}(B)$ s.t. $\mathsf{C}(A) \equiv \neg \bigwedge X$ then $D(A, B) = \top$

It can be observed that for the first three of the above postulates, the consequent is the same, whereas the consequent of the fourth is more stringent about the nature of the attack, and each of the postulates guarantees existence of an argument involved in an attack. Also, the conditions for the first three postulates match the consequent of the corresponding $D1$ postulate. In addition, $D6$ implies $D6'$ and $D6''$.

For an attack function D that satisfies $D1$, $D2$ and $D6$, we show that D is a super-relation of D_{CU} (i.e. $D_{CU} \subseteq D$).

Proposition 7. *Suppose that D satisfies $D1$, $D2$ and $D6$ and that A, B are two arguments in $\mathbb{G}^{\Delta}_{D}$. If $\mathsf{C}(A) \equiv \neg \bigwedge \mathsf{S}(B)$, then $D(A, B) = \top$.*

Proof. The assumption $\mathsf{C}(A) \equiv \neg \bigwedge \mathsf{S}(B)$ entails $\{\mathsf{C}(A)\} \cup \mathsf{S}(B) \vdash \bot$, thus postulate $D6$ applies, producing an argument C such that $\mathsf{C}(A) \vdash \mathsf{C}(C)$ and $D(C, B) = \top$. Note that C need not belong to $\mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$. By $D1$ it follows that $\mathsf{C}(C) \vdash \neg \bigwedge \mathsf{S}(B)$. This in turn means that $\mathsf{C}(A) \equiv \mathsf{C}(C) \equiv \neg \bigwedge \mathsf{S}(B)$. Applying $D2$ yields that $D(A, B) = \top$. □

Next we look at the class of attack functions satisfying the postulates $D1'$, $D2$ and $D6'$. An attack function satisfying these postulates would be a super-relation of D_{DU} (i.e. $D_{DU} \subseteq D$).

Proposition 8. *Suppose that D satisfies $D1'$, $D2$ and $D6'$ and that A, B are two arguments in $\mathbb{G}^{\Delta}_{D}$. If $\mathsf{C}(A) \equiv \neg\phi$ where $\phi \in \mathsf{S}(B)$, then $D(A, B) = \top$.*

Proof. The assumption $\mathsf{C}(A) \equiv \neg\phi$ where $\phi \in \mathsf{S}(B)$ allows us to use postulate $D6'$, producing an argument C such that $\mathsf{C}(A) \vdash \mathsf{C}(C)$ and $D(C, B) = \top$. By $D1'$ it follows that $\mathsf{C}(C) \vdash \neg\psi$ for some $\psi \in \mathsf{S}(B)$. Combining these gives us that $\psi \vdash \phi$, so in order to preserve the minimality of $\mathsf{S}(B)$ it must be that $\phi = \psi$. This further means that $\mathsf{C}(A) \equiv \mathsf{C}(C)$. Applying $D2$ yields that $D(A, B) = \top$. □

For an attack relation D that satisfies $D1''$, $D2$ and $D6''$, we show that D is a super-relation of D_R (i.e. $D_R \subseteq D$).

Proposition 9. *Suppose that D satisfies $D1''$, $D2$ and $D6''$ and that A, B are two arguments in $\mathbb{G}^{\Delta}_{D}$. If $\mathsf{C}(A) \equiv \neg\, \mathsf{C}(B)$, then $D(A, B) = \top$.*

Table 2: Attack functions and the postulates they are characterised by. Each attack function is characterised by the conjunction of the postulates located in the appropriate row, column and (optionally) cell.

	$D1$, $D6$	$D1'$,$D6'$	$D1''$,$D6''$	$D6'''$
$D2'$	D_D	D_{DD}	D_{DR}	—
$D2$	D_{CU} ($D5$)	D_{DU} ($D5'$)	D_R ($D5''$)	—
—	—	—	—	D_U ($D5'''$)

Proof. Using postulate $D6''$ produces an argument C such that $\mathsf{C}(A) \vdash \mathsf{C}(C)$ and $D(C, B) = \top$. By $D1''$ it follows that $\mathsf{C}(C) \vdash \neg\mathsf{C}(B)$. Combining these gives us that $\mathsf{C}(A) \equiv \mathsf{C}(C)$. Applying $D2$ yields that $D(A, B) = \top$. □

We now show that each $D \in \{D_D, D_U, D_{CU}, D_{DU}, D_{DD}, D_R, D_{DR}\}$ can be characterised using the postulates constraining the attack function.

Proposition 10. *The following characterisation results hold and they are summarised in Table 2.*

- $D = D_D$ *iff* D *satisfies* $D1$, $D2'$ *and* $D6$.
- $D = D_{DD}$ *iff* D *satisfies* $D1'$, $D2'$ *and* $D6'$.
- $D = D_{DR}$ *iff* D *satisfies* $D1''$, $D2'$ *and* $D6''$.
- $D = D_{CU}$ *iff* D *satisfies* $D1$, $D2$, $D5$ *and* $D6$.
- $D = D_{DU}$ *iff* D *satisfies* $D1'$, $D2$, $D5'$ *and* $D6'$.
- $D = D_R$ *iff* D *satisfies* $D1''$, $D2$, $D5''$ *and* $D6''$.
- $D = D_U$ *iff* D *satisfies* $D5'''$ *and* $D6'''$.

Proof. We prove the desired results by showing that $D \subseteq D_x$ and $D \supseteq D_x$ iff the appropriate postulates are satisfied by D, for each appropriate D_x. We will omit the proofs for the left-to-right direction as they are straightforward.

(D_D) Suppose D satisfies $D1$, $D2'$ and $D6$. It is easy to see that $D1$ ensures that $D \subseteq D_D$. To show that $D \supseteq D_D$, suppose that $D_D(A, B) = \top$ for some A, B. This means that $\{\mathsf{C}(A)\} \cup \mathsf{S}(B) \vdash \bot$, thus satisfying the conditions of $D6$, which in turn provides an argument C such that $D(C, B) = \top$ and $\mathsf{C}(A) \vdash \mathsf{C}(C)$. By applying $D2'$ we obtain that $D(A, B) = \top$.

(D_{DD}) Suppose D satisfies $D1'$, $D2'$ and $D6'$. It is easy to see that $D1'$ ensures that $D \subseteq D_{DD}$. To show that $D \supseteq D_{DD}$, suppose that $D_{DD}(A, B) = \top$ for some A, B. This means that $\exists \phi \in \mathsf{S}(B)$ s.t. $\mathsf{C}(A) \vdash \neg\phi$ thus satisfying the conditions of $D6'$, which in turn provides an argument C such that $D(C, B) = \top$ and $\mathsf{C}(A) \vdash \mathsf{C}(C)$. By applying $D2'$ we obtain that $D(A, B) = \top$.

(D_{DR}) Suppose D satisfies $D1''$, $D2'$ and $D6''$. It is easy to see that $D1''$ ensures that $D \subseteq D_{DR}$. To show that $D \supseteq D_{DR}$, suppose that $D_{DR}(A, B) = \top$ for some A, B. This means that $\mathsf{C}(A) \vdash \neg\mathsf{C}(B)$, thus satisfying the conditions of $D6''$, which in turn provides an argument C such that $D(C, B) = \top$ and $\mathsf{C}(A) \vdash \mathsf{C}(C)$. By applying $D2'$ we obtain $D(A, B) = \top$.

(D_{CU}) Suppose D satisfies $D1$, $D2$, $D5$ and $D6$. $D1$ and $D5$ guarantee that if $D(A, B) = \top$ then $D_{CU}(A, B) = \top$. For the other direction, we apply Proposition 7 to get $D \supseteq D_{CU}$.

(D_{DU}) Suppose D satisfies $D1'$, $D2$, $D5'$ and $D6'$. To show that $D \subseteq D_{DU}$ suppose that $D(A, B) = \top$. Postulate $D1'$ yields that there is $\phi \in \mathsf{S}(B)$ such that $\mathsf{C}(A) \vdash \neg\phi$ and $D5'$ that there is $\psi \in \mathsf{S}(B)$ such that $\neg\,\mathsf{C}(A) \vdash \psi$. Combining these gives us that $\neg\psi \vdash \neg\phi$, or $\phi \vdash \psi$. The definition of what is an argument enforces minimality, meaning that it must be that $\phi = \psi$, which in turn yields that $\mathsf{C}(A) \equiv \neg\phi$, completing the proof. For the other direction, we apply Proposition 8 to get $D \supseteq D_{DU}$.

(D_R) Suppose D satisfies $D1''$, $D2$, $D5''$ and $D6''$. $D1''$ and $D5''$ together entail that $D \subseteq D_R$. For the other direction, we apply Proposition 9 to get $D \supseteq D_R$.

(D_U) Suppose D satisfies $D5'''$ and $D6'''$. To show that $D \subseteq D_U$, assume that $D(A, B) = \top$. Using $D5'''$ we get that there exists $X \subseteq \mathsf{S}(B)$ such that $\neg\,\mathsf{C}(A) \equiv \bigwedge X$. Therefore, $\exists X \subseteq \mathsf{S}(B)$ such that $\mathsf{C}(A) \equiv \neg \bigwedge X$. For the other direction, from $D_U(A, B) = \top$, $\exists X \subseteq \mathsf{S}(B)$ such that $\mathsf{C}(A) \equiv \neg \bigwedge X$. Therefore, via $D6'''$, we get $D(A, B) = \top$, and hence $D \supseteq D_U$.

□

In this section, we have provided a framework of postulates for the attack relation and shown it can be used for classifying and characterizing instances of attack relation. In the next section, we will use these postulates for the attack relation in order to consider postulates for logical argument graphs. Our postulates $D0 - D4$, and the variants $D2'$ and $D3'$, provide intuitive proposals for necessary constraints on attack relations, and then postulates $D5$ and variants, and $D6$ and variants, provide the extra constraints to characterise them in Proposition 10. If we only have $D0$-$D4$, then there is too much latitude in the constraints they provide, and as a result they permit undesirable attack relations (i.e. if we just seek to satisfy $D0$ to $D4$, then we allow attack relations with counter-intuitive behaviour). We believe that new proposals for attack relations will be made where the underlying logic is classical logic, or another rich formalism such as a description logic, a temporal logic, or a spatial logic, and so we believe having these characterisations as given in this paper will be helpful in defining attack relations for new situations.

4. Postulates for logical argument graphs

The intention behind the argument generator (given in Definition 5) is the generation of an argument graph that can be used with the extension definitions proposed in the literature and reviewed in Section 2. Doing so will yield in each case one or more extensions, each of which is a set of arguments. From the perspective of abstract argumentation we would stop at this point since nothing more can be said about the result using only the information that is encoded in the argument graph. However, since these arguments are logical arguments, it is possible to examine the resulting extensions further.

To this end, we discuss and propose some postulates about the content of extensions. We do not, however, desire to constrain or propose definitions of extensions; this is beyond the scope of our paper and much existing work on alternative extensions is easy to find. What we do desire, here, is to provide an extension-independent set of postulates that are reasonable and succinct.

We introduce some additional phraseology and notation here. A set of formulae Ψ is called ***free*** if $\Psi \subseteq \Delta \setminus \bigcup_{M \in \mathsf{MI}(\Delta)} M$. This definition is obviously dependent on Δ; we omit stating this for clarity and without danger of confusion. An argument A is called a ***free argument*** if $\mathsf{S}(A)$ is free. The set of free arguments belonging to a graph Γ (that has been returned by the generator $\mathbb{G}^{\Delta}_{D}$) is denoted by $\mathsf{FreeArgs}(\Gamma)$ and the set of non-free arguments by $\mathsf{NonFreeArgs}(\Gamma)$. We say that a set of arguments S supports a formula ϕ and write $S \Vdash \phi$ if there is an argument $A \in S$ such that $\mathsf{C}(A) \vdash \phi$.

In order to formulate the postulates we will assume that X is any of the reviewed semantics. We first look at *consistency* postulates. Let D be an attack function and Δ a knowledge base.

($CN1$) $$\bigcup_{A \in \pi_{\mathrm{X}}(\mathbb{G}^{\Delta}_{D})} \mathsf{S}(A) \nvdash \bot$$

($CN2$) $$\bigcup_{A \in S} \mathsf{S}(A) \nvdash \bot, \text{ for all } S \in \mathcal{E}_{\mathrm{X}}(\mathbb{G}^{\Delta}_{D})$$

($CN1'$) $$\bigcup_{A \in \pi_{\mathrm{X}}(\mathbb{G}^{\Delta}_{D})} \mathsf{C}(A) \nvdash \bot$$

($CN2'$) $$\bigcup_{A \in S} \mathsf{C}(A) \nvdash \bot, \text{ for all } S \in \mathcal{E}_{\mathrm{X}}(\mathbb{G}^{\Delta}_{D})$$

Note that these postulates are understood to be universally quantified over Δ. The above postulates are variations of the requirement that certain arguments' supports or claims must be consistent together. The expectation is that once an extension is obtained, then the arguments contained in it present a somehow consistent set of assumptions. Applying this restriction to the supports of the arguments or to their claims, and to the sceptically accepted set of arguments or to all extensions individually, yields the versions of this principle listed above. The reason we provide all four versions is that it is not yet clear whether one form of the postulate is more appropriate than others. For example, consistency postulates similar to *CN1*$'$ and *CN2*$'$ have been proposed in [25] in the context of rule-based argumentation systems and versions of *CN1* and *CN2* have been proposed in [19] for classical logics. It should be clear that *CN2* entails *CN1*, *CN2*$'$ entails *CN1*$'$, *CN1* entails *CN1*$'$ and *CN2* entails *CN2*$'$.

Next we look at the relationship between the set of free arguments and the set of sceptically accepted arguments.

(SC) $$\mathsf{FreeArgs}(\mathbb{G}^{\Delta}_{D}) \subseteq \pi_{\mathrm{X}}(\mathbb{G}^{\Delta}_{D})$$

This postulate encodes our expectation that since free arguments are uncontroversial, they should be asserted in every extension and, hence, in the sceptically accepted set as well.

Finally, we introduce a postulate that captures the expectation that there exists a knowledge base which is inconsistent and for which some of the arguments, that can formed from the knowledge base, are not credulously accepted.

$$(CR) \qquad \exists\Delta \text{ s.t. } \mathsf{NonFreeArgs}(\mathbb{G}_D^\Delta) \neq \emptyset \text{ and } \sigma_{\mathrm{X}}(\mathbb{G}_D^\Delta) \neq \mathsf{Nodes}(\mathbb{G}_D^\Delta)$$

This may seem like a relatively simple postulate, but we will see that it is a good postulate for differentiating the various options we have for argumentation with classical logic. When the postulate fails, it means that for any knowledgebases Δ, and for any argument A, if $\mathsf{S}(A) \subseteq \Delta$, then A is credulously accepted. In other words, for any argument that can be formed from a knowledgebase, there is a preferred extension that contains that argument. So if it does fail for an attack function D and an extension semantics X, then this indicates that the combination of D and X is, in a sense, very credulous.

5. Status of postulates for argument graphs

In this section we examine the status of the postulates proposed in Section 4 under the various reviewed attack functions.

5.1. Postulate SC

First we show that when using any attack function satisfying $D1$, a free argument can neither be attacked by nor attack another argument.

Proposition 11. *Let A be a free argument in $\mathbb{G}_D^\Delta$ and let D be an attack function that satisfies $D1$. Then, there is no $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ such that $D(B, A) = \top$ or $D(A, B) = \top$.*

Proof. Let $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ and suppose by way of contradiction that $(B, A) \in \mathsf{Arcs}(\mathbb{G}_D^\Delta)$, meaning that $D(B, A) = \top$ by Definition 5. As D satisfies $D1$, this means that $\mathsf{C}(B) \vdash \neg \bigwedge \mathsf{S}(A)$. This further entails that $\mathsf{S}(A) \cup \mathsf{S}(B) \vdash \bot$. Let M be the minimal inconsistent subset of $\mathsf{S}(A) \cup \mathsf{S}(B)$. If $M \cap \mathsf{S}(A) \neq \emptyset$ then A cannot be free, thus contradicting our assumption. If $M \cap \mathsf{S}(A) = \emptyset$, then it must be that $M \subseteq \mathsf{S}(B)$, contradicting the consistency of the support of B. In both cases we have a contradiction. The case where A attacks B is proved in an identical way. □

Next we show that for any attack function satisfying $D1$, every complete extension will contain all free arguments in the argument graph.

Proposition 12. *Let S be a complete extension of $\mathbb{G}_D^\Delta$ and let D be an attack function that satisfies $D1$. Then, $S \supseteq \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$.*

Proof. Arguments in $\mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$ have no attacks whatsoever by Proposition 11, so trivially S defends them and therefore must contain them. □

We can use the extension hierarchy to extend the previous result as follows.

Proposition 13. *Let D be an attack function that satisfies $D1$. All reviewed extensions of $\mathbb{G}_D^\Delta$ contain all free arguments and therefore postulate SC is satisfied.*

Proof. Follows directly from Proposition 12 and the extension hierarchy (Figure 1). $\square$

Since all attack functions in $\{D_D, D_U, D_{CU}, D_{DU}, D_{DD}, D_R, D_{DR}\}$ satisfy $D1$, the above result shows that they all satisfy postulate SC as well. Given that such a wide range of options for the attack function satisfy postulate SC, we may regard it as an uncontroversial postulate.

5.2. Postulate CR

Here we present results on the satisfaction of CR. We do this for the categories of attack functions delineated in Table 2. This means that we will use attack function postulates to prove that certain classes of attack functions satisfy or falsify CR.

For reasons to do with the structure of the proofs and economy of space we will address these attack function classes for the stable, semi-stable, preferred and complete semantics first, and then we will examine all these classes together for the grounded and ideal semantics.

5.2.1. Attack functions based on canonical undercuts

In this section we examine the status of CR for the class of attack functions that satisfy $D1$, $D2$ and $D6$, and therefore are super-relations of D_{CU}. We begin by introducing a definition of a set of arguments that is generated by a set of formulae. This is going to be a useful notion in the rest of this section.

Definition 14. *Let $\Phi \subseteq \Delta$ be a consistent set of formulae. The set of arguments S^{Φ} is generated by Φ iff*

$$S^{\Phi} = \{ A \in \mathcal{A} \mid \mathsf{S}(A) \subseteq \Phi \}.$$

It should be clear that for any consistent $\Phi \subseteq \Delta$, it is the case that $S^{\Phi} \subseteq \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$.

Next we show that for every consistent subset of $\Phi \subseteq \Delta$ we can find an admissible set of arguments that contains every argument based on Φ.

Proposition 15. *Suppose that D satisfies $D1$, $D2$ and $D6$, and let $\Phi \subseteq \Delta$ be a consistent set of formulae. Then, the set S^{Φ} is admissible in $\mathbb{G}^{\Delta}_{D}$.*

Proof. First we show that S^{Φ} is conflict-free. Suppose that $A, B \in S^{\Phi}$ and that $D(A, B) = \top$. Then as D satisfies $D1$ this means that $\mathsf{S}(A) \cup \mathsf{S}(B) \vdash \bot$, a contradiction since Φ was assumed consistent.

We now show that S^{Φ} defends itself. Let $A \in S^{\Phi}$ and B an argument that attacks A, meaning that $D(B, A) = \top$. By $D1$, it must be that $\mathsf{S}(A) \cup \mathsf{S}(B) \vdash \bot$, thus $\mathsf{S}(A) \vdash \neg \bigwedge \mathsf{S}(B)$. We take Ψ to be the minimal subset of $\mathsf{S}(A)$ that entails $\neg \bigwedge \mathsf{S}(B)$. Then, $C = \langle \Psi, \neg \bigwedge \mathsf{S}(B) \rangle$ is an argument in $\mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$ and by construction, $C \in S^{\Phi}$. By Proposition 7, we know that $D(C, B) = \top$, completing the proof. $\square$

We can now relate some consistent sets with stable extensions. For this, we require the notion of a maximal consistent subset: For a set of formulae Δ, $\Phi \subseteq \Delta$ is a maximal consistent subset iff Φ is consistent and for all $\Phi' \subseteq \Delta$, if $\Phi \subset \Phi'$, then Φ' is not consistent.

Proposition 16. *Suppose that D satisfies $D1$, $D2$ and $D6$. Let $\Phi \subseteq \Delta$ be a maximal consistent set. Then, S^{Φ} is a stable extension of $\mathbb{G}^{\Delta}_{D}$.*

Proof. Clearly, S^Φ is conflict-free, since it is admissible (Proposition 15). Thus, S^Φ is a stable extension iff S^Φ attacks A for any $A \notin S^\Phi$. For the latter to be the case, it must be that $\mathsf{S}(A) \nsubseteq \Phi$ and from the maximality of Φ it follows that $\Phi \cup \mathsf{S}(A) \vdash \bot$. Let Ψ be the minimal subset of $\mathsf{S}(A)$ such that $\Psi \vdash \neg \bigwedge \Phi$. Then $B = \langle \Psi, \neg \bigwedge \Phi \rangle$ is an argument in $\mathbb{G}_D^\Delta$. It is also clear that $B \notin S^\Phi$ as otherwise the consistency of Φ would be contradicted. Consider the argument $C = \langle \Phi', \bigwedge \Phi \rangle \in S^\Phi$, where $\Phi' \subseteq \Phi$ is the minimal subset that entails the claim. From Proposition 7 we know that $D(B, C) = \top$ (since $\bigwedge \Phi' \equiv \bigwedge \Phi$). By the admissibility of S^Φ it follows that there exists $E \in S^\Phi$ such that $D(E, B) = \top$. By $D1$, it follows that $\mathsf{S}(E) \vdash \neg \bigwedge \mathsf{S}(B)$ and thus $\mathsf{S}(E) \vdash \neg \bigwedge \mathsf{S}(A)$. Taking $\Psi' \subseteq \mathsf{S}(E)$ as the minimal set of formulae that entails $\neg \bigwedge \mathsf{S}(A)$, we obtain $F = \langle \Psi', \neg \bigwedge \mathsf{S}(A) \rangle \in S^\Phi$ and by applying Proposition 7 again it follows that $D(F, A) = \top$, completing the proof. □

We are now in a position to address the status of CR with regards to attack functions that satisfy $D1$, $D2$ and $D6$ and all extension semantics bar the grounded and ideal, which we address separately. Here, we prove that CR is falsified in these circumstances by showing that for any Δ, any argument is credulously accepted.

Proposition 17. *Suppose that D satisfies $D1$, $D2$ and $D6$, and let X be the stable, semi-stable, preferred or complete semantics. Then, for any Δ and any argument $A \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$, $A \in \sigma_X(\mathbb{G}_D^\Delta)$. Therefore, CR is not satisfied.*

Proof. Let $A \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ be an argument and let $\Phi \subseteq \Delta$ be a maximal consistent set of formulae such that $\mathsf{S}(A) \subseteq \Phi$ (such a set always exists by the consistency of supports of arguments). Then Proposition 16 applies confirming that the set of arguments S^Φ is a stable (also semi-stable, preferred and complete) extension. Clearly, $A \in S^\Phi$. Thus, for any Δ and any argument A there is a stable, semi-stable, preferred and complete extension that includes it, completing the proof. □

5.2.2. Attack functions based on direct undercuts

We will look next at the class of attack functions satisfying the postulates $D1'$, $D2$ and $D6'$. An attack function satisfying these postulates would be a super-relation of D_{DU}.

We again use maximally consistent sets to create stable extensions as follows. Note, a similar result to this was given by Cayrol in [16]. The result may also be obtained via the correspondence established by Dung [3] relating stable extensions with extensions of Reiter's default logic where the attack relation is direct undercut, and the correspondence between supernormal default theories and maximal consistent subsets of knowledgebases [26].

Proposition 18. *Suppose that D satisfies $D1'$, $D2$ and $D6'$. If $\Phi \subseteq \Delta$ is a maximally consistent set, then S^Φ is a stable extension in $\mathbb{G}_D^\Delta$.*

Proof. We first show that S^Φ is conflict-free. If not, then there would be $A, B \in S^\Phi$ such that $D(A, B) = \top$, i.e., $D(A, B) = \top$ and by $D1'$, this means that $\mathsf{S}(A) \cup \mathsf{S}(B) \vdash \bot$, contradicting the consistency of Φ.

We next show that S^Φ attacks any argument not in S^Φ. Let $A \in S^\Phi$ and $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta) \setminus S^\Phi$. By construction, the support of B must employ formulae not in Φ. Pick an arbitrary formula $\psi \in \mathsf{S}(B) \setminus \Phi$. As Φ is maximally consistent, it must be that

$\Phi \cup \{\psi\}$ is inconsistent, therefore there is a minimal subset of $\Psi \subseteq \Phi$ such that $\Psi \vdash \neg\psi$. This means that $C = \langle \Psi, \neg\psi \rangle$ is an argument and by construction $C \in S^{\Phi}$. Using Proposition 8 we obtain that $D(C, B) = \top$, completing the proof. □

Using the above results we can show that the postulate CR is falsified.

Proposition 19. *Suppose that D satisfies $D1'$, $D2$ and $D6'$, and let X be the stable, semi-stable, preferred or complete semantics. Then, for any Δ and any argument $A \in \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$, $A \in \sigma_X(\mathbb{G}^{\Delta}_{D})$. Therefore, CR is not satisfied.*

Proof. Let $\Phi \subseteq \Delta$ be a maximal consistent set of formulae such that $\mathsf{S}(A) \subseteq \Phi$. Then Proposition 18 applies confirming that the set of arguments S^{Φ} is a stable (also semi-stable, preferred and complete) extension. Clearly, $A \in S^{\Phi}$. Thus, for any Δ and any argument A there is a stable, semi-stable, preferred and complete extension that includes it, completing the proof. □

5.2.3. Attack functions based on rebuttals

Here we examine the status of CR in the context of attack functions that are super-relations of rebuttals, i.e., those that satisfy $D1''$, $D2$ and $D6''$. For these attack functions, we show that we can construct a stable extension.

Proposition 20. *Suppose D satisfies $D1''$, $D2$ and $D6''$, and $A \in \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$. Then there exists a stable extension of $\mathbb{G}^{\Delta}_{D}$ that contains A.*

Proof. A set of arguments generated by a maximally consistent subset of Δ will be an admissible set. However it will not necessarily be a stable extension since it partitions the set of all arguments on the basis of their supports and not on the basis of their claims. It is for this reason that we need the following definition.

Let Φ be a set of formulae such that:

(a) There exists $\phi \in \Phi$ such that $\phi = \mathsf{C}(A)$.
(b) If $\phi \in \Phi$ then there exists $B \in \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$ such that $\phi = \mathsf{C}(B)$.
(c) For any $\phi, \psi \in \Phi$, it is the case that $\{\phi, \psi\} \nvdash \bot$.
(d) There is no Ψ which satisfies the above conditions and is a proper superset of Φ.

Such a set must clearly always exist. We then define $S \subseteq \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$ to be the set of arguments whose claims are in Φ: $B \in S$ if and only if $\mathsf{C}(B) \in \Phi$.

First we show that S is conflict-free. Assuming the opposite, we obtain two arguments $B, C \in S$ such that $D(B, C) = \top$. Applying $D1''$ we get that $\mathsf{C}(B) \vdash \neg\mathsf{C}(C)$, meaning that $\{\mathsf{C}(B), \mathsf{C}(C)\} \vdash \bot$ contradicting condition (c) above.

Now, we show that for any argument $B \in \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D}) \setminus S$, there is an argument $C \in S$ such that $D(C, B) = \top$. Before doing that we prove that Φ is closed under entailment. Suppose $\phi \in \Phi$ and that $\phi \vdash \psi$ and that $\psi \notin \Phi$. This can only be because there is $\chi \in \Phi$ such that $\{\psi, \chi\} \vdash \bot$, or that $\psi \vdash \neg\chi$. But then $\phi \vdash \neg\chi$ contradicting the assumption that $\phi, \chi \in \Phi$.

Next, we see that from the assumption that $B \notin S$ we obtain that $\mathsf{C}(B) \notin \Phi$ and, by condition (d) above, that there is $C \in S$ such that $\{\mathsf{C}(B), \mathsf{C}(C)\} \vdash \bot$. Thus, $\mathsf{C}(C) \vdash \neg\mathsf{C}(B)$. However, Φ is closed under entailment as shown above, therefore there must be an argument $C' \in S$ such that $\mathsf{C}(C') \equiv \neg\mathsf{C}(B)$. Applying Proposition 9 yields $D(C', B) = \top$, and thus, S attacks B. □

Proposition 21. *Suppose that D satisfies $D1''$, $D2$ and $D6''$, and let X be the stable, semi-stable, preferred or complete semantics. Then, for any Δ and any argument $A \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$, $A \in \sigma_X(\mathbb{G}_D^\Delta)$. Therefore, CR is not satisfied.*

Proof. Proposition 20 applies confirming that the set of arguments S is a stable (also semi-stable, preferred and complete) extension. Clearly, $A \in S$. Thus, for any Δ and any argument A there is a stable, semi-stable, preferred and complete extension that includes it, completing the proof. □

5.2.4. Attack functions based on undercuts

Finally, we consider the status of CR in the case of the class of attack functions that satisfy $D1$, $D2$ and $D6'''$. An attack function satisfying these postulates would be a superset of D_U.

Similarly to previous sections, we use maximally consistent sets of formulae to generate stable extensions as follows.

Proposition 22. *Suppose that D satisfies $D1$, $D2$ and $D6'''$. If $\Phi \subseteq \Delta$ is a maximally consistent set, then S^Φ is a stable extension in $\mathbb{G}_D^\Delta$.*

Proof. We first show that S^Φ is conflict-free. If not, then there are $A, B \in S^\Phi$ such that $D(A, B) = \top$, and by $D1$, this means that $\mathsf{S}(A) \cup \mathsf{S}(B) \vdash \bot$, contradicting the consistency of Φ.

We next show that S^Φ attacks any argument not in S^Φ. Let $A \in S^\Phi$ and $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta) \setminus S^\Phi$. By construction, the support of B must employ formulae not in Φ. As Φ is maximally consistent, it must be that $\Phi \cup \mathsf{S}(B)$ is inconsistent. Thus, there is a set $\Psi \subseteq \mathsf{S}(B)$ and a minimal set $X \subseteq \Phi$ such that $C = \langle X, \neg \bigwedge \Psi \rangle$ is an argument. By construction, it is the case that $C \in S^\Phi$. Using $D6'''$, we obtain that $D(C, B) = \top$, completing the proof. □

Proposition 23. *Suppose that D satisfies $D1$, $D2$ and $D6'''$, and let X be the stable, semi-stable, preferred or complete semantics. Then, for any Δ and any argument $A \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$, $A \in \sigma_X(\mathbb{G}_D^\Delta)$. Therefore, CR is not satisfied.*

Proof. Let $\Phi \subseteq \Delta$ be a maximal consistent set of formulae such that $\mathsf{S}(A) \subseteq \Phi$. Then Proposition 22 applies confirming that the set of arguments S^Φ is a stable (also semi-stable, preferred and complete) extension. Clearly, $A \in S^\Phi$. Thus, for any Δ and any argument A there is a stable, semi-stable, preferred and complete extension that includes it, completing the proof. □

5.2.5. The Grounded and Ideal Extensions

We now turn to the ideal and grounded extensions and examine these in the context of all four attack function classes we have seen before. What distinguishes the ideal and grounded extension is that they are 'sceptical' extensions by definition, that is to say, there is exactly one ideal and one grounded extension at all times, and so some arguments will be excluded.

In the case of an attack function that satisfies $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$, the ideal and grounded extensions can only contain free arguments. To prove this, we first show that any non-free argument has an attacker.

Proposition 24. *Let D be an attack function that satisfies $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$. If $A \in \mathsf{NonFreeArgs}(\mathbb{G}_D^\Delta)$ then there is an argument $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ such that $D(B, A) = \top$.*

Proof. As A is non-free, there exists a minimal inconsistent set $M \in \mathsf{MI}(\Delta)$ such that $M \cap \mathsf{S}(A) \neq \emptyset$. It is easy to show that in each case above, there exists an argument $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ such that $D(B, A) = \top$. For the case D satisfying $D1$ and $D6$, $M \cap \mathsf{S}(A) \neq \emptyset$ implies there is a C such that $\mathsf{S}(C) \subseteq M$ and $\mathsf{C}(C) \cup \mathsf{S}(A) \vdash \bot$. Therefore, $\mathsf{C}(C) \vdash \neg \bigwedge \mathsf{S}(A)$. Therefore, there is a B such that $\mathsf{S}(B) \subseteq \mathsf{S}(C)$ and $\mathsf{C}(C) \vdash \mathsf{C}(B)$ and $\mathsf{C}(B) \equiv \neg \bigwedge \mathsf{S}(A)$. Therefore, by the application of Proposition 13, $D(B, A) = \top$. For the case D satisfying $D1'$ and $D6'$, $M \cap \mathsf{S}(A) \neq \emptyset$ implies there is a $\phi \in \mathsf{S}(A)$ such that $M \setminus \{\phi\} \vdash \neg\phi$. Therefore, there is a B such that $\mathsf{S}(B) = M \setminus \{\phi\}$ and $\mathsf{C}(B) = \neg\phi$. Therefore, by the application of Proposition 8, $D(B, A) = \top$. For the case D satisfying $D1$ and $D6'''$, $M \cap \mathsf{S}(A) \neq \emptyset$ implies there is an $X \subseteq \mathsf{S}(A)$ and an argument B such that $\mathsf{S}(B) = M \setminus X$ and $\mathsf{C}(B) \equiv \neg \bigwedge X$. Therefore, by $D6'''$, $D(B, A) = \top$. □

We also obtain the following weaker result for an attack function that satisfies $D2$ and $D1''$ and $D6''$.

Proposition 25. *Let D be an attack function that satisfies $D2$ and $D1''$ and $D6''$. If $A \in \mathsf{NonFreeArgs}(\mathbb{G}_D^\Delta)$ then there are arguments $B, C \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ such that $\mathsf{S}(B) = \mathsf{S}(A)$ and $D(C, B) = \top$ and $D(B, C) = \top$.*

Proof. As A is non-free, there exists a minimal inconsistent set $M \in \mathsf{MI}(\Delta)$ such that $M \cap \mathsf{S}(A) \neq \emptyset$. Therefore, there exists an argument B such that $\mathsf{S}(A) = \mathsf{S}(B)$ and $\mathsf{C}(B) = \bigwedge \mathsf{S}(A)$. Since $\mathsf{S}(A) \subseteq M$, there also exists an argument C such that $\mathsf{S}(C) \subseteq M$ and $\mathsf{C}(C) = \neg \bigwedge \mathsf{S}(A)$. Therefore, by the application of Proposition 9, $D(C, B) = \top$ and $D(B, C) = \top$. □

Now we can prove that the ideal and the grounded extensions are equal to the set of free arguments for an attack function that satisfies $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$.

Proposition 26. *Let D be an attack function that satisfies $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$. Then the ideal and grounded extensions are equal to the set of free arguments.*

$$\sigma_{gr}(\mathbb{G}_D^\Delta) = \sigma_{id}(\mathbb{G}_D^\Delta) = \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$$

Proof. We begin by showing that the intersection of all preferred extensions does not contain any non-free arguments, or $\pi_{\mathrm{pr}}(\mathbb{G}_D^\Delta) \cap \mathsf{NonFreeArgs}(\mathbb{G}_D^\Delta) = \emptyset$. Suppose A is a non-free argument. From Propositions 17, 19, and 23, we know that there exists a preferred extension $P_A \in \mathcal{E}_{\mathrm{pr}}(\mathbb{G}_D^\Delta)$ such that $A \in P_A$. From Proposition 24 we obtain that there is an argument $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ such that $D(B, A) = \top$. By applying Proposition 17, Proposition 19 or Proposition 23, once again, we know that there is a preferred extension $P_B \in \mathcal{E}_{\mathrm{pr}}(\mathbb{G}_D^\Delta)$ such that $B \in P_B$. As both P_A and P_B are by definition conflict-free, $A \notin P_A \cap P_B$ and therefore $A \notin \pi_{\mathrm{pr}}(\mathbb{G}_D^\Delta)$. Thus, $\pi_{\mathrm{pr}}(\mathbb{G}_D^\Delta) \subseteq \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$.

It is known that for any argument graph Γ, the grounded extension is a subset of the ideal extension which, in turn, is a subset of the intersection of all preferred extensions [13].

$$\sigma_{\mathrm{gr}}(\Gamma) \subseteq \sigma_{\mathrm{id}}(\Gamma) \subseteq \pi_{\mathrm{pr}}(\Gamma)$$

Combining Proposition 13 and the above result gives us

$$\mathsf{FreeArgs}(\mathbb{G}^{\Delta}_{D}) \subseteq \sigma_{\mathrm{gr}}(\mathbb{G}^{\Delta}_{D}) \subseteq \sigma_{\mathrm{id}}(\mathbb{G}^{\Delta}_{D}) \subseteq \pi_{\mathrm{pr}}(\mathbb{G}^{\Delta}_{D}) \subseteq \mathsf{FreeArgs}(\mathbb{G}^{\Delta}_{D}),$$

or in other words,

$$\sigma_{\mathrm{gr}}(\mathbb{G}^{\Delta}_{D}) = \sigma_{\mathrm{id}}(\mathbb{G}^{\Delta}_{D}) = \mathsf{FreeArgs}(\mathbb{G}^{\Delta}_{D}).$$

□

Finally, we address the status of CR for the ideal and the grounded extensions.

Proposition 27. *Let D be an attack function that satisfies $D2$ and one of the following pairs of postulates: $D1$ and $D6$, or $D1'$ and $D6'$, or $D1''$ and $D6''$, or $D1$ and $D6'''$. Then, the postulate CR is satisfied for the grounded and ideal extensions.*

Proof. Suppose Δ is inconsistent. This means that $\mathsf{MI}(\Delta) \neq \emptyset$, and thus, $\mathsf{NonFreeArgs}(\mathbb{G}^{\Delta}_{D}) \neq \emptyset$. For D that satisfies $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$, it will be the case that $\mathsf{FreeArgs}(\mathbb{G}^{\Delta}_{D}) \subset \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$ and therefore (by Proposition 26) $\sigma_{\mathrm{gr}}(\mathbb{G}^{\Delta}_{D}) = \sigma_{\mathrm{id}}(\mathbb{G}^{\Delta}_{D}) \neq \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$. For D that satisfies $D2$, $D1''$ and $D6''$, it will be the case that there are arguments $A, B \in \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$ such that $D(B, A) = \top$ and $D(A, B) = \top$ (following from Proposition 25). Therefore, $A \notin \sigma_{\mathrm{id}}(\mathbb{G}^{\Delta}_{D})$ or $B \notin \sigma_{\mathrm{id}}(\mathbb{G}^{\Delta}_{D})$. Hence, $\sigma_{\mathrm{id}}(\mathbb{G}^{\Delta}_{D}) \neq \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$. Also since $\sigma_{\mathrm{gr}}(\mathbb{G}^{\Delta}_{D}) \subseteq \sigma_{\mathrm{id}}(\mathbb{G}^{\Delta}_{D})$, $\sigma_{\mathrm{gr}}(\mathbb{G}^{\Delta}_{D}) \neq \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$. □

The CR postulate has allowed us to differentiate the options for argumentation with classical logic where all arguments that can be constructed from the knowledgebase are considered in the argument graph. For the credulous semantics (i.e. stable, semi-stable, preferred, and complete semantics) we get that every non-free argument is contained in at least one, but not all extensions, whereas for the sceptical semantics (i.e. grounded and ideal semantics) with the attack function satisfying $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$, we get that the extensions contain no non-free arguments. An interpretation of these results is that the credulous semantics is perhaps too credulous and the sceptical semantics is perhaps too sceptical. Furthermore, these results suggest that a different defeat function or extension semantics should be used or that an alternative to the argument generator (Definition 5) should be used. Alternatively, priorities over formulae, and thereby over arguments, can be harnessed (e.g. [27, 28]).

5.3. Consistency

In this section we look at the status of the consistency postulates. For the attack functions that only have free arguments in the grounded and the ideal extensions, it is easy to see that *CN1* will hold.

Proposition 28. *Let D be an attack function that satisfies $D2$ and one of the following pairs $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$. Then, the postulate CN1 is satisfied for the grounded and ideal extensions.*

Proof. As shown in Proposition 26, the ideal and the grounded extensions are equal to the set of free arguments. Let $\Phi = \bigcup_{A \in \mathsf{FreeArgs}(\mathbb{G}^{\Delta}_{D})} \mathsf{S}(A)$ and suppose that $\Phi \vdash \bot$. This means that there is $M \in \mathsf{MI}(\Delta)$ such that $M \subseteq \Phi$. But this means that there is at least one argument $A \in \mathsf{FreeArgs}(\mathbb{G}^{\Delta}_{D})$ such that $M \cap \mathsf{S}(A) \neq \emptyset$, contradicting the definition of free arguments. □

We can generalise this result for the other extensions as follows.

Proposition 29. *Let D be an attack function that satisfies $D2$ and one of the following pairs $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$. Then, for any of the reviewed extension semantics X,*

$$\pi_X(\mathbb{G}_D^\Delta) = \mathsf{FreeArgs}(\mathbb{G}_D^\Delta).$$

Proof. We have already shown this result for the ideal and grounded extensions in Proposition 26. We do this here for the stable, semi-stable, preferred and complete semantics.

From the extension hierarchy, we know that

$$\mathcal{E}_{\mathrm{st}}(\mathbb{G}_D^\Delta) \subseteq \mathcal{E}_{\mathrm{ss}}(\mathbb{G}_D^\Delta) \subseteq \mathcal{E}_{\mathrm{pr}}(\mathbb{G}_D^\Delta) \subseteq \mathcal{E}_{\mathrm{co}}(\mathbb{G}_D^\Delta).$$

Thus, the sceptically accepted sets of arguments will obey an inequality in the opposite direction.

$$\pi_{\mathrm{st}}(\mathbb{G}_D^\Delta) \supseteq \pi_{\mathrm{ss}}(\mathbb{G}_D^\Delta) \supseteq \pi_{\mathrm{pr}}(\mathbb{G}_D^\Delta) \supseteq \pi_{\mathrm{co}}(\mathbb{G}_D^\Delta)$$

From Proposition 12, we know that $\pi_{\mathrm{co}}(\mathbb{G}_D^\Delta) \supseteq \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$. Thus it suffices to show that $\pi_{\mathrm{st}}(\mathbb{G}_D^\Delta) \subseteq \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$. Suppose that a non-free argument A is a member of $\pi_{\mathrm{st}}(\mathbb{G}_D^\Delta)$. This means that A belongs to every stable extension of $\mathbb{G}_D^\Delta$. As shown in Proposition 24, there must be an argument $B \in \mathsf{Nodes}(\mathbb{G}_D^\Delta)$ such that $D(B, A) = \top$. By Propositions 17, 19 and 23, it follows that there is a stable extension S_B containing B. But this leads to a contradiction, since $A, B \in S_B$ and S_B is a stable extension and, therefore, conflict-free. □

Thus, it is simple to show that *CN1* is satisfied for attack functions that satisfy $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$.

Proposition 30. *Let D be an attack function that satisfies $D2$ and either $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$. Then, the postulate CN1 is satisfied for the stable, semi-stable, preferred and complete extensions.*

Proof. Let $\mathsf{Core}(\Delta) = \bigcup_{M \in \mathsf{MI}(\Delta)} M$ and let $\mathsf{Free}(\Delta) = \Delta \setminus \mathsf{Core}(\Delta)$. Hence, $\mathsf{Free}(\Delta) \nvdash \bot$. Also, $A \in \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$ iff $\mathsf{S}(A) \subseteq \mathsf{Free}(\Delta)$. So, $\bigcup_{A \in \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)} \mathsf{S}(A) \subseteq \mathsf{Free}(\Delta)$. Hence, $\bigcup_{A \in \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)} \nvdash \bot$. From Proposition 29, $\pi_{\mathrm{X}}(\mathbb{G}_D^\Delta) = \mathsf{FreeArgs}(\mathbb{G}_D^\Delta)$. Therefore, $\bigcup_{A \in \pi_{\mathrm{X}}(\mathbb{G}_D^\Delta)} \nvdash \bot$. □

For an attack function that satisfies $D2$, and either $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$, it is clear that as *CN1* is satisfied, so is *CN1′* by implication. Whereas for an attack function that satisfies $D1''$, $D2$ and $D6''$, we show that postulates *CN1* and *CN1′* are not satisfied using the following examples.

Example 31. *Let $\Delta = \{a \wedge b, \neg a \wedge c\}$. Hence, $\langle\{a \wedge b\}, b\rangle$ and $\langle\{\neg a \wedge c\}, c\rangle$ are arguments in $\pi_X(\mathbb{G}_D^\Delta)$ for the reviewed semantics when an attack function that satisfies $D1''$, $D2$ and $D6''$, and clearly $\{a \wedge b, \neg a \wedge c\} \vdash \bot$. Hence, postulate CN1 is not satisfied for an attack function that satisfies $D1''$, $D2$ and $D6''$.*

Example 32. *Consider the* $\Delta = \{a \wedge \neg c \wedge d, (\neg a \vee \neg b) \wedge c \wedge d, b \wedge c \wedge \neg d\}$*. These formulae are pairwise inconsistent, and so each argument from this knowledgebase has only one premise. Now, consider the following arguments which can be formed from* Δ*.*

$$A_1 = \langle \{a \wedge \neg c \wedge d\}, a\rangle$$
$$A_2 = \langle \{(\neg a \vee \neg b) \wedge c \wedge d\}, \neg a \vee \neg b\rangle$$
$$A_3 = \langle \{b \wedge c \wedge \neg d\}, b\rangle$$

For each $A_i \in \{A_1, A_2, A_3\}$*, there is no argument* B *from* Δ *such that* B *is a rebuttal of* A_i*. Therefore,* A_1*,* A_2*, and* A_3*, belong to each preferred extension of* Δ*, and so* $\{A_1, A_2, A_3\} \subseteq \pi_X(\mathbb{G}^{\Delta}_{D})$*. However,* $\{\mathsf{C}(A_1), \mathsf{C}(A_2), \mathsf{C}(A_3)\} \vdash \bot$*. Therefore, postulate CN1′ is not satisfied for an attack function that satisfies* $D1''$*,* $D2$ *and* $D6''$*.*

Next we turn our attention to postulates *CN2* and *CN2′*. It is easy to see that due to Proposition 29 and the fact that there is exactly one grounded and one ideal extension, these extension semantics will always satisfy *CN2* and *CN2′*, when the attack function satisfies $D2$ and one of $D1$ and $D6$, or $D1'$ and $D6'$, or $D1$ and $D6'''$. For an attack function that satisfies $D1''$ and $D6''$, we can use Example 31 to show postulate *CN2* is not satisfied and Example 32 to show postulate *CN2′* is not satisfied. In what follows, we examine the remaining extension semantics. We begin by showing a general result regarding postulate $D3'$.

Proposition 33. *Let* D *be an attack function that satisfies* $D3'$*. Then any complete extension of* $\mathbb{G}^{\Delta}_{D}$ *is closed under sub-arguments (where* A *is a sub-argument of* B *when* $\mathsf{S}(A) \subseteq \mathsf{S}(B)$*).*

Proof. Suppose that S is a complete extension, $A \in S$, B is a sub-argument of A and that C is an argument that attacks B. As $\mathsf{S}(B) \subseteq \mathsf{S}(A)$ it follows from $D3'$ that $D(C, A) = \top$. But, as S is a complete extension it must defend A and therefore defend B as well. Thus, $B \in S$. □

With Proposition 33 at our disposal we can approach the status of *CN2* and *CN2′*. Next, we show that attack functions satisfying $D2$, $D1'$, $D6'$ and $D3'$, satisfy both of these postulates. Such attack functions include D_{DD} and D_{DU}.

Proposition 34. *Let* D *be an attack function that satisfies* $D2$*,* $D1'$*,* $D6'$ *and* $D3'$*. Then, for any of the reviewed extension semantics X, postulates CN2 and CN2′ hold.*

Proof. Having already discussed the status of these postulates regarding the grounded and the ideal extensions, we only need to address the case of the stable, semi-stable, preferred and complete extensions. Once again we will make use of the extension hierarchy and prove that these postulates are satisfied for any complete extension. As all other extensions are also complete, the result follows. Also, we will only show that *CN2* is satisfied, as in that case, *CN2′* will be satisfied by implication. Suppose then that $S \subseteq \mathsf{Nodes}(\mathbb{G}^{\Delta}_{D})$ is a complete extension.

Let $\Phi = \bigcup_{A \in S} \mathsf{S}(A)$. Aiming for a contradiction, assume that $\Phi \vdash \bot$, or in other words, that there exists $M \in \mathsf{MI}(\Delta)$ such that $M \subseteq \Phi$. Clearly, this means that there exists an argument $A \in S$ such that $M \cap \mathsf{S}(A) \neq \emptyset$. Pick a formula $\psi \in M \cap \mathsf{S}(A)$ and consider the argument $B = \langle M \setminus \{\psi\}, \neg\psi\rangle$. Using Proposition 8, we conclude that

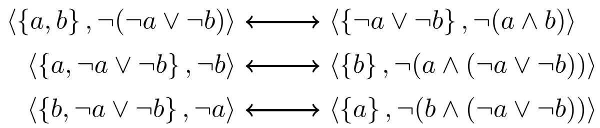


Figure 3: Argument graph for Example 35

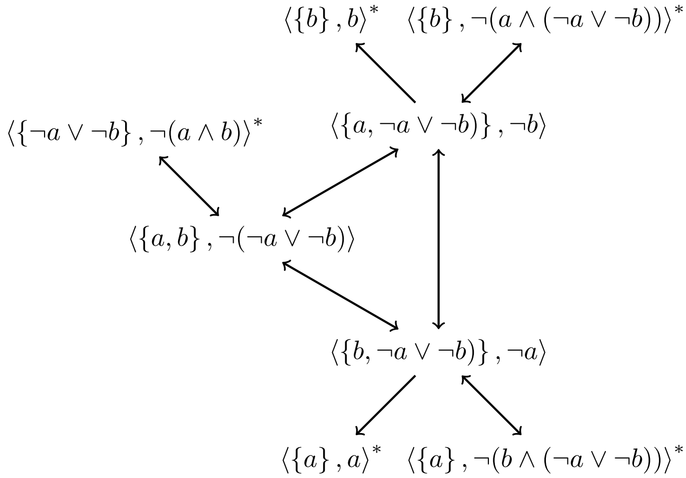


Figure 4: Argument graph for Example 36

$D(B,A)=\top$ and that, since S is a complete extension, S must defend A. Thus, there exists an argument $C\in S$ such that $D(C,B)=\top$. From $D1'$ follows that there exists $\chi\in\mathsf{S}(B)$, and therefore $\chi\in M$, such that $\mathsf{C}(C)\vdash\neg\chi$. There also exists an argument $E=\langle\Psi,\neg\chi\rangle$ such that $\Psi\subseteq\mathsf{S}(C)$ is the minimal subset that entails $\neg\chi$. By Proposition 8, it follows that $D(E,B)=\top$ and as E is a sub-argument of C it must be that $E\in S$, as shown in Proposition 33. As $M\subseteq\Phi$ and $\chi\in M$ there must exist an argument $F\in S$ such that $\chi\in\mathsf{S}(F)$. Another application of Proposition 8 yields that $D(E,F)=\top$, contradicting the assumption that S is conflict-free. $\square$

The remaining attack functions, namely D_D, D_{CU}, D_U, D_R and D_{DR}, do not satisfy *CN2* nor *CN2'*. We provide counterexamples below.

Example 35. *Consider the knowledge base $\Delta=\{a,b,\neg a\vee\neg b\}$ and suppose we are using the attack function D_{CU}. It can be seen that there is a stable extension containing all arguments, e.g., on the left column. Such an extension violates both CN2 and CN2'.*

Example 36. *In a similar way to the last example, we will use the knowledge base $\Delta=\{a,b,\neg a\vee\neg b\}$ again with the attack function D_U. We present the the graph in Figure 4: Here, there is a readily observable stable extension that violates CN2 and CN2', namely the five arguments marked with *.*

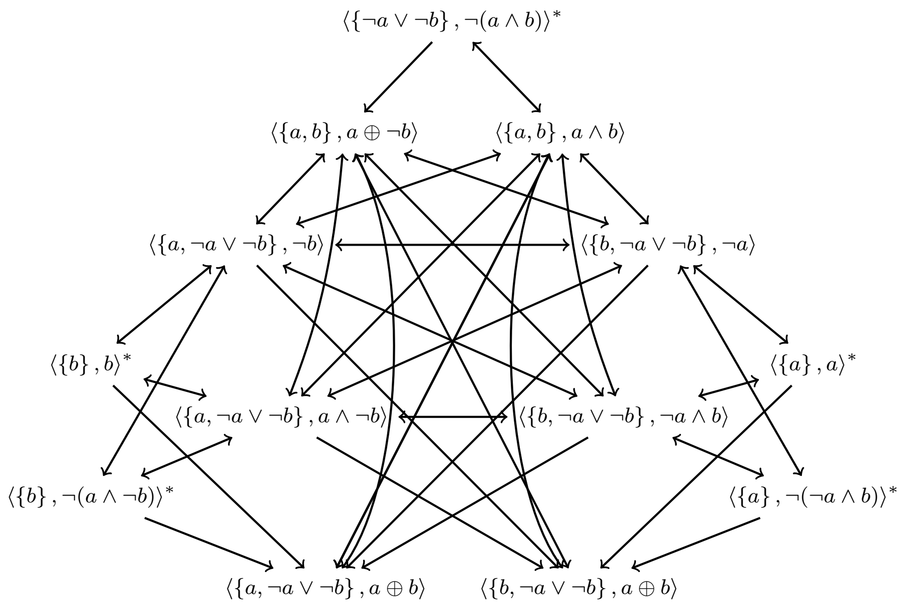


Figure 5: Argument graph for Example 37

$$\begin{array}{rcl}
\langle \{a \wedge b\}, \neg(a \wedge \neg b)\rangle & \longleftrightarrow & \langle \{a \wedge \neg b\}, a \wedge \neg b\rangle \\
\langle \{a \wedge b\}, a \wedge b\rangle^* & \longleftrightarrow & \langle \{a \wedge \neg b\}, \neg(a \wedge b)\rangle \\
\langle \{a \wedge b\}, (a \wedge b) \vee (\neg a \wedge \neg b)\rangle & \longleftrightarrow & \langle \{a \wedge \neg b\}, (a \wedge \neg b) \vee (\neg a \wedge b)\rangle \\
\langle \{a \wedge b\}, b\rangle & \longleftrightarrow & \langle \{a \wedge \neg b\}, \neg b\rangle^*
\end{array}$$

Figure 6: Argument graph for Example 38

Example 37. *We again use the knowledge base* $\Delta = \{a, b, \neg a \vee \neg b\}$ *with the attack function* D_D*. The relevant part of the graph is shown in Figure 5. We use* $\oplus$ *to denote concisely the XOR operation, i.e.,* $\phi \oplus \psi \equiv (\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$*. It can be seen that the five arguments marked with* * *form part of a stable extension, while the union of their supports is inconsistent, as well as the union of their claims. Thus* D_D *fails to satisfy CN2 and CN2*$'$*.*

Example 38. *We use the knowledge base* $\Delta = \{a \wedge b, a \wedge \neg b\}$ *and show that when using* D_R*, the postulates CN2 and CN2*$'$ *are not satisfied. For this, we get the argument graph in Figure 6. There are several stable extensions containing both the arguments marked with* **. Every such stable extension violates both CN2 and CN2*$'$*.*

Note, Cayrol in [16] has also given an example that shows that D_{DR} violates *CN2*.

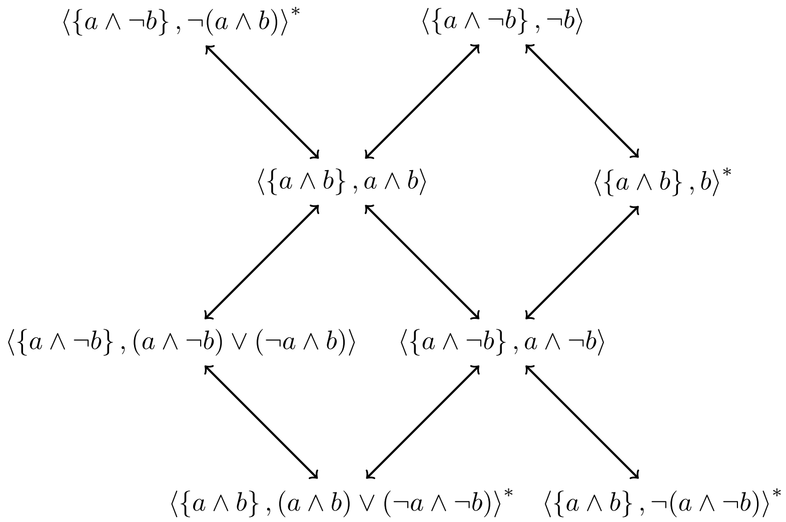


Figure 7: Argument graph for Example 39

Example 39. *We show that D_{DR} violates CN2 with the knowledge base $\Delta = \{a \wedge b, a \wedge \neg b\}$. The argument graph is given in Figure 7. It should be clear that the set of arguments marked with * is a stable extension that violates CN2 as well as CN2′.*

In conclusion, we can state the following. Postulates *CN1* and *CN1′* are satisfied by all combinations of extension semantics with all attack functions considered in this paper that satisfy $D3$. Postulates *CN2* and *CN2′* are satisfied by all attack functions considered in this paper that satisfy $D3$ in the case of the grounded and ideal extensions and only by D_{DD} and D_{DU} in the case of the complete, preferred, semi-stable and stable semantics. As D_{DU} is the most widely considered defeat relation in the literature, it is interesting that the *CN2* and *CN2′* postulates hold with credulous semantics. However, the failure of these postulates with other defeat relations is an issue that may be interpreted as a weakness of the defeat relation or of the specific semantics, and perhaps raising the need for alternatives to be identified. Another response is that it is not the defeat relation and dialectical semantics that should be responsible for ensuring that all the premises used in the winning arguments are consistent together. Rather, it could be argued that checking that the premises used are consistent together should be the responsibility of something external to the defeat relation and dialectical semantics, and so knowing whether the *CN2* and *CN2′* postulates hold or not influences what external mechanisms are required for checking. Furthermore, checking consistency of premises of sets of arguments may be part of the graph construction process. For instance, in Garcia and Simari's proposal for dialectical trees [29], there are constraints on what arguments can be added to the tree based on consistency with the premises of other arguments in the tree.

6. Discussion

In this paper we have investigated the use of classical logic for constructing argument graphs. The first component of this approach is an attack relation and we have explored various desirable properties for them in Section 3. Using these properties, we have been able to classify and characterise several concrete attack relations. We then proceeded to propose postulates that express desirable requirements concerning the logical content of extensions of argument graphs constructed using classical logic in Section 4. Now, we examine related work and discuss the ways that we can extend this research.

Classical logic has been considered previously as a logic for constructing argument graphs. Wooldridge, Dunne and Parsons, in [30] investigate the complexity of some decision problems on classical logic argumentation graphs with the direct undercut attack relation. Although very interesting because of the computational issues raised, [30] does not examine the properties of the content of extensions. In [27], Amgoud and Cayrol propose an argumentation framework with preferences, which uses classical logic. The focus of the paper is on the use of preferences and, therefore, the consequences of using classical logic are not investigated. In [19], Amgoud and Besnard have similar goals to ours and propose a consistency condition which we have drawn upon for our postulates. They examine special cases of knowledge bases and symmetric attack relations and whether consistency is satisfied in this context. Then in [20], Amgoud and Besnard extend this analysis by showing a correspondence between the maximal consistent subsets of a knowledgebase and the maximal conflict-free sets of arguments. Cayrol, in [16], presents results similar to Proposition 18 and Example 38. However, in all these cases, the focus is on consistency and few other properties of the logical content of extensions are studied. In addition, there is a lack of consideration of the range of attack relations.

Postulates regarding properties of logical argumentation graphs have been investigated before. Caminada and Amgoud [25, 18], and Caminada [31], propose a consistency postulate and a closure postulate (which we discuss further below). This is done, however, in the context of a rule-based language as opposed to classical logic, and using a fixed attack relation.

Similarly, Martínez, García and Simari, in [32], modify Dung's definitions of acceptability to include the notion of warrant, and in the process formulate a condition similar to $D3'$, which they call *conflict inheritance*. In contrast to our work, [32] stays at the level of abstract argument graphs and does not extend to the object-language level.

Given that we want our framework to capture existing proposals, we have allowed for arguments with syntactically different supports to be differentiated. Hence, this means that we have specified that two arguments are equivalent if they have identical support and logically equivalent claims. However, the framework can be adapted so that arguments A and B are equivalent if they have logically equivalent support and logically equivalent claims. This then raises the need to consider further postulates in future work such as the following.

($D3^{*}$) if $D(A,B) = \top$ and $\mathsf{S}(B) \equiv \mathsf{S}(C)$ then $D(A,C) = \top$

($D3^{**}$) if $D(A,B) = \top$ and $\mathsf{S}(C) \vdash \bigwedge \mathsf{S}(B)$ then $D(A,C) = \top$

With regards to further work, several questions remain open. An important question is whether there exist postulates other than those we examine here, that are interesting to study. As mentioned above, Caminada and Amgoud have proposed a *closure* postulate in the context of rule-based argumentation systems. Such a principle could be encoded as follows in our approach.

($CL1$) if $\bigcup_{A \in \pi_X(\mathbb{G}^{\Delta}_{D})} \mathsf{S}(A) \vdash \phi$ then $\pi_X(\mathbb{G}^{\Delta}_{D}) \Vdash \phi$

($CL2$) if $\bigcup_{A \in S} \mathsf{S}(A) \vdash \phi$ then $S \Vdash \phi$, for all $S \in \mathcal{E}_X(\mathbb{G}^{\Delta}_{D})$

($CL1'$) if $\bigcup_{A \in \pi_X(\mathbb{G}^{\Delta}_{D})} \mathsf{C}(A) \vdash \phi$ then $\pi_X(\mathbb{G}^{\Delta}_{D}) \Vdash \phi$

($CL2'$) if $\bigcup_{A \in S} \mathsf{C}(A) \vdash \phi$ then $S \Vdash \phi$, for all $S \in \mathcal{E}_X(\mathbb{G}^{\Delta}_{D})$

Here, the intuition is that if the arguments in an extension carry the information required to prove a particular formula, then there should be an argument in the extension whose claim entails that formula. Once again there are several variations possible on this principle, as seen above. Variations of the postulates $CL1'$ and $CL2'$ have appeared in [31, 18]. Here, we introduce $CL1$ and $CL2$ as well, as natural extensions in the context of classical logic. It is easy to see that $CL1$ entails $CL2$, $CL1'$ entails $CL2'$, $CL1'$ entails $CL1$ and $CL2'$ entails $CL2$.

Another question for future work is the choice of definition for generating logical argument graphs. The definition for the argument generator (Definition 5) which we use is a natural choice, but it would be useful to investigate alternative definitions to account for proposals such as the argument tree definition in [6, 9].

Recently, a number of interesting developments of abstract argumentation have been proposed. These build on Dung's proposal to provide more sophisticated modelling of argumentation, such as value-based argumentation framework (which allow for the moral values of the audience to be taken into account) [33], bipolar abstract frameworks (which allow for support relations to also be included) [34], argumentation frameworks with weighted attacks [35, 36], extended frameworks where arguments can express preference between other arguments [37, 38]. It would be interesting, in future work, to investigate the instantiation of these frameworks with classical logic, and at the same time identify new postulates to constrain and justify these instantiations.

Finally, we have identified that for some attack relations, when Φ is a maximal consistent set, and S^{Φ} is the set of arguments generated by Φ (as given in Definition 12), then S^{Φ} is a stable extension. So another interesting question is whether the converse holds.

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