An elementary formulation of Riemann's Zeta function

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Abstract

We rewrite Riemann's Zeta function as a sum over the primes. Each term of the sum is a product that depends only on the summation index (a prime) and the primes following it.

Definitions. *Let* $\{p_k\}$ *be the sequence of the primes,*

$$\mathcal{Z}_i(s) = \prod_{k=1}^{i} (1 - p_k^{-s})^{-1},$$

$$\mathcal{S}_i(s) = \sum_{k=1}^{i} \left(p_k^{-s} \prod_{j=k}^{i} (1 - p_j^{-s})^{-1} \right)$$

and

$$\mathcal{I}_i = \{s \in \mathbb{C} \,|\, p^{-s} = -1 \textit{ for some prime } p \le p_i\}.$$

Explicitly

$$\mathcal{I}_i = \left\{ s \in \mathbb{C} \,\middle|\, \Re(s) = 0 \wedge \left(\exists k \in \mathbb{Z}, j \in \{1, \cdots, i\} \,:\, \Im(s) = \frac{(1+2k)\pi}{\ln p_j} \right) \right\}.$$

Theorem. *For any* $i \in \mathbb{N}$ *and* $s \in \mathbb{C} \setminus \mathcal{I}_i$, *we have* $Z_i(s) = 1 + S_i(s)$.

Proof: We proceed by induction.
Base case: For $i = 1$ we have

$$\mathcal{S}_1(s) = p_1^{-s}(1 - p_1^{-s})^{-1} \Rightarrow$$

$$1 + \mathcal{S}_1(s) = 1 + p_1^{-s}(1 - p_1^{-s})^{-1} = \frac{1 - p_1^{-s} + p_1^{-s}}{1 - p_1^{-s}} = \frac{1}{1 - p_1^{-s}} = \mathcal{Z}_1(s).$$

Induction hypothesis: Assume

$$\mathcal{Z}_i(s) = 1 + \mathcal{S}_i(s).$$

By adding and subtracting p_{i+1}^{-s} to the r.h.s. we obtain

$$\begin{aligned} \mathcal{Z}_i(s) &= \left(p_{i+1}^{-s} + \mathcal{S}_i(s)\right) + \left(1 - p_{i+1}^{-s}\right) \Rightarrow \\ (1 - p_{i+1}^{-s})^{-1} \mathcal{Z}_i(s) &= (1 - p_{i+1}^{-s})^{-1} \left(p_{i+1}^{-s} + \mathcal{S}_i(s)\right) + (1 - p_{i+1}^{-s})^{-1}(1 - p_{i+1}^{-s}) \Rightarrow \end{aligned}$$

$$
\begin{aligned}
\mathcal{Z}_{i+1}(s) &= (1-p_{i+1}^{-s})^{-1}\left(p_{i+1}^{-s}+\sum_{k=1}^{i}\left(p_k^{-s}\prod_{j=k}^{i}(1-p_j^{-s})^{-1}\right)\right)+1\\
&= (1-p_{i+1}^{-s})^{-1}p_{i+1}^{-s}+\sum_{k=1}^{i}\left(p_k^{-s}\prod_{j=k}^{i+1}(1-p_j^{-s})^{-1}\right)+1\\
&= \sum_{k=1}^{i+1}\left(p_k^{-s}\prod_{j=k}^{i+1}(1-p_j^{-s})^{-1}\right)+1 \Rightarrow
\end{aligned}
$$

$$\mathcal{Z}_{i+1}(s) = 1+\mathcal{S}_{i+1}(s).$$

□

Corollary. *For $\Re(s)>1$, Riemann's Zeta function can be written as*

$$\zeta(s) = 1+\sum_{k=1}^{\infty} a_k(s)p_k^{-s}$$

where $a_k(s) = \prod_{j=k}^{\infty}(1-p_j^{-s})^{-1}$.

Proof: The result follows directly from noting that Euler's product formula can be expressed as

$$\zeta(s) = \lim_{i\to\infty} \mathcal{Z}_i$$

and from the previous theorem. □

In this formulation, the Riemann's Zeta function is expressed as a sum over the primes, where each term is a product with factors depending only on the summation index (a prime) and the primes following it. The formulation is interesting also because of its similarity with the original definition of the Zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, the factors $a_k(s)$ effectively being corrections that account for all the (non-prime) naturals whose prime factor decomposition starts with p_k.

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