COMP0114 Inverse Problems in Imaging. PreTest

1 Introduction

This is a self-test for you to see if you have the correct mathematical background to take the course : COMP0114 "Inverse Problems in Imaging"

2 Test

2.1 Complex Numbers

In this section we define the *imaginary unit* i as

$$\mathrm{i} = \sqrt{-1}$$

1. Show that $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, and use this to prove *de Moivre's Theorem*

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

   for integer n.
2. If $z = \cos\theta + \mathrm{i}\sin\theta$, find the values of

$$z + \frac{1}{z}, \quad z^2 + \frac{1}{z^2}, \quad z^n + \frac{1}{z^n}, \quad z^n - \frac{1}{z^n}$$

   in terms of θ.
3. Show that
$$(1+\mathrm{i})^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + \mathrm{i}\sin\frac{n\pi}{4}\right)$$
4. Show that
$$\left(1+\mathrm{i}\sqrt{3}\right)^n + \left(1-\mathrm{i}\sqrt{3}\right)^n = 2^{n+1}\cos\frac{n\pi}{3}$$

2.2 Calculus

1. If $f(x) = x\ln x - x$ What is the derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$?
2. If $f(x) = x\mathrm{e}^x$ what is the (indefinite) integral $F(x) = \int_{y=-\infty}^{x} f(y)\mathrm{d}y$. (I.e. $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$)?
3. If we define $r = \sqrt{x^2+y^2+z^2}$ and $f(r) = \frac{1}{r}$ what is the *gradient*

$$\nabla f(r) := \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} ?$$

2.3 Fourier Transforms

In this section we defined the one-dimensional Fourier Transform $\mathcal{F}_{x\to k}$ of a function $f(x)$ as

$$F(k) = \mathcal{F}_{x\to k}[f] := \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x.$$

1. What is the Fourier Transform of the *Rectangle function* ?

$$\mathsf{Rect}(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2} \end{cases}$$

2. What is the Fourier Transform of the *Triangle function* ?

$$\mathsf{Triangle}(x) = \begin{cases} 1+x & \text{if } -1 \le x \le 0 \\ 1-x & \text{if } \quad 0 \le x \le 1 \\ 0 & \text{if } \quad |x| > 1 \end{cases}$$

3. If we define the *convolution* of two functions $f(x)$ and $g(x)$ as

$$h(x) = f(x) * g(x) := \int_{y=-\infty}^{y=\infty} f(y)g(x-y)\mathrm{d}y$$

then prove the *convolution theorem* :

$$H(k) = \mathcal{F}_{x\to k}h(x), \quad G(k) = \mathcal{F}_{x\to k}g(x), \quad F(k) = \mathcal{F}_{x\to k}hfx), \qquad \Rightarrow \qquad H(k) = G(k)F(k)$$

2.4 Vector Calculus

In this section we use the notation $\times$ for the vector product, and $\cdot$ for the scalar product.

1. Find an equation for the plane perpendicular to the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ that passes through the point $Q = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$.
   What is the distance from the origin to this plane ?
2. Prove that (a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ and (b) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$.
3. Prove that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$.

2.5 Differential Geometry

1. A particle moves along a curve such that its position at time t is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-t} \\ 2\cos 3t \\ 2\sin 3t \end{pmatrix}$$

Derive expressions for a) the velocity, b) the acceleration of this particle.

2. Sketch the space curve $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\cos t \\ 3\sin t \\ 4t \end{pmatrix}$ and find

(a) the *unit tangent vector* $\hat{\mathbf{T}}$ (i.e. the tangent to the curve such that $|\hat{\mathbf{T}}| = 1$),
(b) the *principle normal* $\hat{\mathbf{N}}$ (i.e. the component of the acceleration normal to the tangent and such that $|\hat{\mathbf{N}}| = 1$),
(c) the *binormal direction* $\hat{\mathbf{B}}$ perpendicular to both $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$,
(d) the value of the *curvature* κ satisfying

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t} = \kappa\hat{\mathbf{B}},$$

(e) the value of the *torsion* τ satisfying

$$\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}t} = -\tau\hat{\mathbf{N}}.$$

2.6 Differential Equations

1. Show that any twice differentiable function of one variable $f(s)$ is a solution of the *wave equation*

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

where c is a constant if either $s = x - ct$ or $s = x + ct$.

2. Show that the *Green's function* $G(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\mathrm{e}^{-\frac{x^2}{4\kappa t}}$ solves the *diffusion equation*

$$\kappa\frac{\partial^2 G}{\partial x^2} = \frac{\partial G}{\partial t}$$

for all values of x and t except where both $x = 0$ and $t = 0$. For a fixed value $\kappa = 1$, sketch the graph of $G(x,t)$ as a function of x and $t > 0$ and deduce the value of $G(x,t)$ in the limit as $x \to 0$ and $t \to 0$.