1 Introduction

This is a self-test for you to see if you have the correct mathematical background to take the course: COMP0115 “Geometry of Images.”

2 Test

2.1 Complex Numbers

In this section we define the imaginary unit $i$ as

$$i = \sqrt{-1}$$

1. Show that $e^{i\theta} = \cos \theta + i \sin \theta$, and use this to prove de Moivre’s Theorem

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

for integer $n$.

2. If $z = \cos \theta + i \sin \theta$, find the values of

$$z + z^{-1}, \quad z^2 + z^{-2}, \quad z^n + z^{-n}, \quad z^n - z^{-n}$$

in terms of $\theta$.

3. Show that

$$(1 + i)^n = 2^{n/2}\left( \cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right)$$

4. Show that

$$\left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n = 2^{n+1}\cos \frac{n\pi}{3}$$

2.2 Calculus

1. If $f(x) = x \ln x - x$ What is the derivative $\frac{df}{dx}$?

2. If $f(x) = xe^x$ what is the (indefinite) integral $F(x) = \int_{y=-\infty}^{x} f(y) dy$. (I.e. $\frac{dF}{dx} = f(x)$)?

3. If we define $r = \sqrt{x^2 + y^2 + z^2}$ and $f(r) = \frac{1}{r}$ what is the gradient

$$\nabla f(r) := \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$
2.3 Fourier Transforms

In this section we defined the one-dimensional Fourier Transform \( \mathcal{F}_{x \to k} \) of a function \( f(x) \) as

\[
F(k) = \mathcal{F}_{x \to k}[f] := \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} f(x)e^{-ikx}dx.
\]

1. What is the Fourier Transform of the Rectangle function?

\[
\text{Rect}(x) = \begin{cases} 
1 & \text{if } |x| \leq \frac{1}{2} \\
0 & \text{if } |x| > \frac{1}{2} 
\end{cases}
\]

2. What is the Fourier Transform of the Triangle function?

\[
\text{Triangle}(x) = \begin{cases} 
1 + x & \text{if } -1 \leq x \leq 0 \\
1 - x & \text{if } 0 \leq x \leq 1 \\
0 & \text{if } |x| > 1 
\end{cases}
\]

3. If we define the convolution of two functions \( f(x) \) and \( g(x) \) as

\[
h(x) = f(x) * g(x) := \int_{y=\infty}^{y=-\infty} f(y)g(x-y)dy
\]

then prove the convolution theorem:

\[
H(k) = \mathcal{F}_{x \to k}h(x), \quad G(k) = \mathcal{F}_{x \to k}g(x), \quad F(k) = \mathcal{F}_{x \to k}hx, \quad \Rightarrow \quad H(k) = G(k)F(k)
\]

2.4 Vector Calculus

In this section we use the notation \( \times \) for the vector product, and \( \cdot \) for the scalar product.

1. Find an equation for the plane perpendicular to the vector \( \mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \) that passes through the point \( Q = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \).

What is the distance from the origin to this plane?

2. Prove that (a) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \) and (b) \( (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \).

3. Prove that \( (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \).
2.5 Differential Geometry

1. A particle moves along a curve such that its position at time $t$ is given by

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
e^{-t} \\
2 \cos 3t \\
2 \sin 3t
\end{pmatrix}
$$

Derive expressions for a) the velocity, b) the acceleration of this particle.

2. Sketch the space curve

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
3 \cos t \\
3 \sin t \\
4t
\end{pmatrix}
$$

and find

(a) the unit tangent vector $\hat{T}$ (i.e. the tangent to the curve such that $|\hat{T}| = 1$),

(b) the principle normal $\hat{N}$ (i.e. the component of the acceleration normal to the tangent and such that $|\hat{N}| = 1$),

(c) the binormal direction $\hat{B}$ perpendicular to both $\hat{T}$ and $\hat{N}$,

(d) the value of the curvature $\kappa$ satisfying

$$\frac{d\hat{T}}{dt} = \kappa \hat{B},$$

(e) the value of the torsion $\tau$ satisfying

$$\frac{d\hat{B}}{dt} = -\tau \hat{N}.$$

2.6 Differential Equations

1. Show that any twice differentiable function of one variable $f(s)$ is a solution of the wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

where $c$ is a constant if either $s = x - ct$ or $s = x + ct$.

2. Show that the Green's function $G(x, t) = \frac{1}{\sqrt{4\pi \kappa t}} e^{-\frac{x^2}{4\kappa t}}$ solves the diffusion equation

$$\kappa \frac{\partial^2 G}{\partial x^2} = \frac{\partial G}{\partial t}$$

for all values of $x$ and $t$ except where both $x = 0$ and $t = 0$. For a fixed value $\kappa = 1$, sketch the graph of $G(x, t)$ as a function of $x$ and $t > 0$ and deduce the value of $G(x, t)$ in the limit as $x \to 0$ and $t \to 0$. 