Modeling and Statistical Inference : Exercises

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Theory III

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0.1 Exercises

1. The arrival of patients at a doctor's surgery may be regarded as a Poisson process. Twenty patients arrive in an hour; what are the Poisson probabilities for the number of arrivals in a given 3-minute period.

2. A 500-page book contains 50 misprints. What is the probability that there will be more than two misprints on a particular page?

3. Calculate the Poisson probabilities for $\mu = \frac{1}{2}$, and calculate the first few binomial probabilities for $n = 50, a = 0.01$. Why is $0.99^{50} \simeq \mathrm{e}^{-\frac{1}{2}}$?

4. The road accidents in a certain area occur at the rate of one every two days. Calculate the probability of $0, 1, 2, \ldots 6$ accidents per week in this district. What is the most likely number of accidents per week ? How many days in a week are expected to be free of accidents ?

5. Find the probability function for the length of time we have to wait before the first occurence of a random event which happens on average once every six seconds, and compare it with the waiting time before a six appears on a die thrown once a second. Repeat for the time before the second occurence.

0.2 Exercises

1. Find the Moment Generating Function (MGF) in the following cases
   a) When 0, 1 are equally likely
   b) When -1,0,1 are equally likely
   c) For the negative exponenetial function $P(x) = k\mathrm{e}^{-kx}$
   d) For the function $P(x) = 2x \, for \, 0 \leq x \leq 1$
   e) For the binomial probabilities with parameters n, a
   f) For the Poisson probabilities with parameter μ

2. Find $E[x]$, $E[y]$,$E[x+y]$,$E[xy]$, for the following ranges and PDFS :
   a) $\Omega =$ the unit square, $P(x,y) = 4(1 - x - y + xy)$
   b) $\Omega =$ the unit square, $P(x,y) = 6(x-y)^2$
   c) $\Omega =$ the quarter plane x, y positive, $P(x,y) = \mathrm{e}^{-(x+y)}$
   d) $\Omega =$ a quarter circle radius 3, x, y positive, $P(x,y)$ a quarter cone radius 3, with volume 1

3. If $y = mx + c$ and $M(t)$ is the MGF for x, what is the MGF for y ?

0.3 Exercises

1. You are told that one person in three is left-handed. You need to find a left-handed person to ask questions for a survey. You ask seven people before finding a left-handed person. Should this cause you to doubt the original statement?

2. Light bulbs are supposed to have a life whose PDF is Normal with mean 900 hours and standard deviation 80 hours. You test a bulb to destruction and find that it lasts a) 1060 hours, b) 700 hours. Do either of these tests cast doubt on the original statement?

3. I.Q.'s are supposed to form a Normal population with mean 100 and standard deviation 15. A group of five people have I.Q.s $\{95, 105, 124, 130, 133\}$. Are these a particularly unrepresentative group?

4. In an experiment with two dice, the following are observed

Score	2	3	4	5	6	7	8	9	10	11	12
Frequency	3	11	11	9	12	16	19	11	8	5	2

is this result indication of any unfairness in the dice?

5. In a survey of the maintenance habits of car owners, 1890 car tyres were inspected. Examine the following data for evidence as to whether or not the spare wheel is exchanged regularly with other wheels on the car.

	new	part-worn	worn	badly worn
wheels in use	350	902	288	73
spare wheel	107	79	59	32

0.4 Exercises

1. A roulette wheel has spaces that are alternately red and black. The ball starts in one of the spaces and when the wheel is spun it moves round before coming to rest again in one of the spaces. The wheel is biased. It is observed that when it starts in a red space it ends in a red space 40% of the time, but when it starts in a black space it ends in a black space 70% of the time.

   a) write down the probability matrix with "red" in the first row and column.
   b) If the initial probability vector is $\begin{pmatrix}0.5\\0.5\end{pmatrix}$, find the probability vector for the outcome of the second spin
   c) If the probability vector at a certain stage is $\begin{pmatrix}\frac{3}{8}\\\frac{5}{8}\end{pmatrix}$, find the probability vector for the previous spin.

2. A general probability matrix P and a probability vector $\boldsymbol{u}$ can be written

$$\mathsf{P} = \begin{pmatrix}1-a & b\\ a & 1-b\end{pmatrix} , \quad \boldsymbol{u} = \begin{pmatrix}u\\v\end{pmatrix}$$

   Show that the vector $\mathsf{P}\boldsymbol{u}$ has components whose sum is unity.

3. P_1, P_2, P_3, are three probability matrices :

$$\mathsf{P}_1 = \begin{pmatrix}\frac{1}{2} & 0\\ \frac{1}{2} & 1\end{pmatrix} \quad \mathsf{P}_2 = \begin{pmatrix}\frac{1}{6} & \frac{3}{4}\\ \frac{5}{6} & \frac{1}{4}\end{pmatrix} \quad \mathsf{P}_3 = \begin{pmatrix}\frac{1}{5} & \frac{3}{5}\\ \frac{4}{5} & \frac{2}{5}\end{pmatrix}$$

   a) Find the products of any pair of matrices and show that the result is a probability matrix.
   b) Show that P_1^3 and P_3^3 are probability matrices.
   c) Find the eigenvalues and eigenvectors of P_1, P_2, P_3
   d) Show that the product of the eigenvalues is equal to the determinant of the matrix in each case.

4. If $\mathsf{P} = \begin{pmatrix}1-a & b\\ a & 1-b\end{pmatrix}$, and $\mathsf{Q} = \begin{pmatrix}1-c & d\\ c & 1-d\end{pmatrix}$ are stochastic (with $0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1, 0 \le d \le 1$) show by direct multiplication that PQ is stochastic.

5. if the two eigenvalues of a 2×2 stochastic matrix are equal, what is the form of the matrix ? What are its eigenvectors ?

6. For the three matrices in question 3, verify the eigendecomposition $\mathsf{P} = \mathsf{E}\Lambda\mathsf{E}^{-1}$, where E has the eigenvectors as columns, and Λ is diagonal with the eigenvalues as the diagonal entries. Thus find the form of P^∞.

7. Find the eigenvalues and eigenvectors of the matrix $\mathsf{P} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and show that P^n does not tend to a limit as $n \to \infty$.

8. Two bags contain two black and two white balls respectively. A transition consists of interchanging a ball drawn at random from each bag.
a) Identify the three possible states of the system, and write down the 3×3 transition matrix for the system.
b) What is the probability that the first bag contains one ball of each colour after three transitions?
c) Find the eigenvalues and eigenvectors of this matrix and determine the limiting probabilty vector for the system.

9. The probability of a team winning a match is 0.6 and of drawing it is 0.3 if the previous match was won. if the previous match was drawn the corresponding probabilities are 0.2 and 0.6 and if it was lsot 0.2 and 0.4. Find the transition matrix and hence the probability of winning or drawing any particular match in the distant future if the probabilities remain the same.