2-Tree probe interval graphs have a large obstruction set

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Abstract

Probe interval graphs (PIGs) are used as a generalization of interval graphs in physical mapping of DNA. $G = (V, E)$ is a probe interval graph (PIG) with respect to a partition $(P, N)$ of $V$ if vertices of $G$ correspond to intervals on a real line and two vertices are adjacent if and only if their corresponding intervals intersect and at least one of them is in $P$; vertices belonging to $P$ are called probes and vertices belonging to $N$ are called non-probes. One common approach to studying the structure of a new family of graphs is to determine if there is a concise family of forbidden induced subgraphs. It has been shown that there are two forbidden induced subgraphs that characterize tree PIGs. In this paper we show that having a concise forbidden induced subgraph characterization does not extend to 2-tree PIGs; in particular, we show that there are at least 62 minimal forbidden induced subgraphs for 2-tree PIGs.

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1. Introduction

The probe interval graph (PIG) model was introduced and used in the human genome project as a more powerful and flexible tool than an interval graph model for the assembly of contigs in the physical mapping of DNA [16–18]. Small fragments of DNA, called clones, are taken from multiple copies of the same genome, and the problem is to reconstruct the
arrangement of these clones. In other words, physical mapping of DNA has the goal of reconstructing relative positions of clones along the original DNA. This problem of finding whether pairs of clones overlap in a long DNA strand can be modeled by an interval graph if we are interested in overlap information between each pair of clones; vertices represent clones and two vertices are adjacent if and only if the clones overlap. In the PIG model, we can use any subset of clones, called probes, and test for overlap information between a pair of clones if and only if at least one clone is a probe. This allows flexibility, since not all DNA fragments need to be known at the time of the construction of a PIG, as is the case in an interval graph model. Thus, the PIG model can be used in real-time applications with growing data sets by generating incremental DNA maps which provide useful information for each further step. We now give a formal definition of this model.

A graph $G = (V, E)$ is an intersection graph of a collection of sets if the vertices of $G$ represent those sets and two distinct vertices are adjacent in $G$ if and only if their corresponding sets have a non-empty intersection. An interval graph is an intersection graph of a family of intervals on a real line. $G = (V, E)$ is a PIG with respect to a partition $(P, N)$ of $V$ if vertices of $G$ correspond to intervals on a real line and two vertices are adjacent if and only if their corresponding intervals intersect and at least one of them is in $P$; vertices belonging to $P$ are called probes and vertices belonging to $N$ are called non-probes.

There has been a lot of interest in PIGs lately. They have been shown to be weakly triangulated, and thus perfect [12]. The hierarchy of graph classes in the neighborhood of PIGs has been described, and also a new class generalizing chordal graphs to probe chordal graphs has been introduced in analogy to the generalization of interval graphs to PIGs [2,1,7,6]. There exist two recognition problems for PIGs. The first recognition problem asks about recognizing, finding and representing possible layouts of the intervals of a PIG with a given partition of its vertices; we refer to this problem as the GP recognition problem (stands for given partition). The second recognition problem for PIGs asks if a given graph is a PIG without knowing a partition of its vertices; we refer to this problem as the non-GP recognition problem. Polynomial time algorithms for the GP recognition problem have recently appeared; in particular, an $O(n^2)$ algorithm [8] and an $O(n + m \log n)$ algorithm [10] have been developed, where $n$ is the number of vertices and $m$ is the number of edges of a graph. An application of an algorithm for constructing a probe interval model occurred in recognizing circular arc graphs [9]. The non-GP recognition problem is unresolved and is attracting considerable attention.

In studying the structure of a new family of graphs a common approach is to determine when the graphs can be characterized by a succinct set of forbidden induced subgraphs. We use the term FISC to refer to the forbidden induced subgraph characterization for a family of graphs. In the case of PIGs, as will be seen in the next section, Sheng [15] has taken the first step in this direction by studying FISCs for acyclic PIGs, with or without a given vertex partition. In particular, Sheng solved the non-GP recognition problem for tree PIGs by showing that tree PIGs can be characterized by two forbidden induced subgraphs. This result gives hope that there is a succinct FISC for chordal PIGs, or even PIGs themselves. As a first step in this direction, it is expected that 2-trees, a natural generalization of trees defined in the next section, will have a succinct FISC. Surprisingly, this is not the case. In this paper we show that the FISC for 2-tree PIGs contains at least 62 graphs. Thus, it is very unlikely that there is a concise FISC for PIGs, or even chordal PIGs.
2. Preliminaries

All graphs in this paper are simple. We denote a graph by $G = (V, E)$, where $V$ is the vertex set of $G$ and $E$ is the edge set of $G$. We also denote $V$ of $G$ by $V(G)$ and $E$ of $G$ by $E(G)$. For a subset $U$ of $V$, we denote by $G(U)$ the subgraph of $G$ induced by the vertices of $U$, and write $G(U) \subseteq G$. The standard definitions of path length and path size are used, representing the number of edges and the number of vertices on the path. The distance between vertices $u$ and $v$ in $G$, denoted by $d(u, v)$, is the number of edges on a shortest $uv$-path. A graph consisting of a path $P_k$ of size $k$ and a vertex $u \notin V(P_k)$ which is universal to $V(P_k)$ is called a $k$-fan. If $K_j$ is a complete graph on $j$ vertices, a graph $G$ is obtained by $K_j$-bonding of graphs $G_1$ and $G_2$ if vertices of a $K_j$ of $G_1$ are identified with the vertices of a $K_j$ of $G_2$. The set $N(x) = \{v \in V \mid vx \in E\}$ is the neighborhood of vertex $x$, and $N[x] = N(x) \cup \{x\}$ is the closed neighborhood of $x$. An asteroidal triple (AT) is an independent set of three vertices in $G$ such that there exists a path between each pair of vertices that avoids the neighborhood of the third vertex. A graph without an AT is called AT-free. Vertices of an AT are called AT-vertices. We say that a collection of sets $\{X, Y, Z\}$ is an asteroidal collection (AC) if for all $x \in X$, for all $y \in Y$, and for all $z \in Z$, $\{x, y, z\}$ is an AT. Each of the sets $X, Y,$ and $Z$ is called an asteroidal set (AS). We defined PIGs in the previous section. An interval representation, $I = \{I_v \mid v \in V\}$, of a PIG $G = (V, E_G)$ is a set of intervals of a real line demonstrating that $G$ is a PIG; clearly, the intersection graph $H = (V, E_H)$ of an interval representation $I$ of a PIG $G$ is an interval graph, and $E_G \subseteq E_H$.

We now give a recursive definition of a $k$-tree $G$: a complete graph on $k$ vertices, $K_k$, is a $k$-tree; if $G$ is a $k$-tree, then so is $G'$ formed from $G$ by adding a new vertex adjacent to all vertices in a $K_k$ in $G$. Thus, a tree is a 1-tree.

As mentioned previously, Sheng has taken the first step in giving a FISC for a restricted family of PIGs, namely trees. In particular, she proved the following [15].

**Theorem 1 (Sheng [15]).** Let $T = (V, P, E)$ be a tree with $P \subset V$ and $N = V \setminus P$. $T$ is a PIG with respect to $P$ if and only if $T(N)$ is an independent set and $T$ has no induced subgraph isomorphic to graph $G_i$, $i = 1, 2, \ldots, 5$ in Fig. 1, with darkened vertices in $P$ and circled vertices in $P$ or $N$.

**Theorem 2 (Sheng [15]).** Let $T = (V, E)$ be a tree. Then $T$ is a PIG if and only if $T$ has no induced subgraph isomorphic to graph $G_4$ or $G_6$ in Fig. 1.

**Lemma 1 (Sheng [15]).** At least one AT-vertex of an AT in a PIG must be a non-probe.
In order to provide the foundation for our search for a FISC for 2-tree PIGs, we now present some general structure results of PIGs.

3. Some structure of PIGs

An immediate consequence of Lemma 1 is the following simple corollary:

Corollary 1. At least one AS of an AC of a PIG G must contain all non-probes. Thus, at least one AS of a PIG must be an independent set.

Proof. Otherwise, there exist probe vertices \( x \in X, y \in Y, \) and \( z \in Z \) such that \( \{x, y, z\} \) is an all probe AT contradicting Lemma 1. □

Claim 1. Let \( \{x, y, z\} \) be an AT of a PIG G. Then in an interval representation of G, no interval corresponding to a vertex in \( \{x, y, z\} \) properly contains an interval corresponding to another vertex in \( \{x, y, z\} \).

Proof. Denote by \( I_x, I_y, \) and \( I_z \) intervals corresponding to \( x, y, \) and \( z, \) respectively, in an interval representation of G. Without loss of generality, let \( I_x \subseteq I_y \). If either \( x \) or \( y \) are probes, then by the definition of a PIG, they must be adjacent, contradicting \( \{x, y, z\} \) being an independent set. Thus, the interesting case is when \( x, y \in N \). Since \( x \in N \), every neighbor of \( x \) in G must be a probe. Thus, the neighbor of \( x \) on every \( x,z \)-path in G must be a probe. Since by the definition of a PIG, the interval corresponding to the neighbor of \( x \) on every \( x,z \)-path must overlap \( I_x \), and since every neighbor of \( x \) in G is a probe, and since \( I_x \subseteq I_y \), every neighbor in G of \( x \) is adjacent in G to \( y \), and thus \( y \) hits every \( x,z \)-path in G contradicting \( \{x, y, z\} \) being an AT in G. □

Claim 2. If \( \{x, y, z\} \) is an all non-probe AT of a PIG G, with intervals \( I_x = [x_1, x_2] \), \( I_y = [y_1, y_2] \), and \( I_z = [z_1, z_2] \) corresponding to \( x, y, \) and \( z, \) in an interval representation I of G, and if one of these intervals, say \( I_i, i \in \{x, y, z\}, \) is properly contained in the interval \( [a, b] \), where \( a \) is the minimum of the left-most vertices and \( b \) is the maximum of the right-most vertices of the other two intervals, then there exists a non-probe internal vertex \( v \) of a j, k-path such that \( I_i \subset I_v \), where \( j, k \in \{x, y, z\} \backslash \{i\}, j \neq k \).

Proof. Without loss of generality assume that \( I_y \in [x_1, z_2] \). Since the same argument applies to all arrangements of \( I_x, I_y, \) and \( I_z \) on the real line, we will consider only one of them, namely let \( I_x \cap I_y \neq \emptyset \) and let \( I_z \) not overlap \( I_y \). Since by Claim 1 no interval in \( \{I_x, I_y, I_z\} \) properly contains another, without loss of generality let \( x_1 < y_1 < x_2 < y_2 \). First, let \( y_2 < z_1 \), and consider an \( x,z \)-path \( P_{x,z} \) in G that avoids \( N(y) \). Since \( x, z \in N \), the neighbor of \( x \) in G and the neighbor of \( z \) in G on \( P_{x,z} \) must both be probe and thus their corresponding intervals cannot overlap \( I_y \). Since the union of the corresponding intervals in I of the vertices of \( P_{x,z} \) overlaps \( I_y \), and since \( P_{x,z} \) avoids \( N(y) \) in G, there must exist a non-probe internal vertex \( v \) of \( P_{x,z} \) such that \( I_v \supset I_y \). Similarly, if \( z_1 < y_2 \), then there must exist a non-probe internal vertex \( v \) on an \( x,z \)-path such that \( I_v \supset I_y \). □
We now give a structural result on a $P, N$ partition in a PIG with an AT.

**Claim 3.** In every AT of a PIG $G = (V, E_G)$ there must exist a non-probe AT vertex $u$ such that there exists a path between the other two AT-vertices that avoids $N(u)$ and has a non-probe internal vertex.

**Proof.** Let $I = \{I_v \mid v \in V\}$ be an interval representation of $G$. Let $H = (V, E_H)$ be the intersection graph of $I$. Let $\{x, y, z\}$ be an AT of $G$ and without loss of generality let $z \in N$. Since $H$ is an interval graph, $H$ does not have any ATs, so $\{x, y, z\}$ is not an AT of $H$, and thus we have the following two cases to consider regarding $\{x, y, z\}$ in $H$: (1) $xy \notin E_H$; (2) $xy \in E_H$ and for every $x, y$-path $P_{x,y}$ there exists a vertex $u \in V(P_{x,y})$ such that $uz \in E_H$.

(1) First we consider the case when $xy \notin E_H$. Since $xy \notin E_G$, this means that $x, y \in N$, and $I_x \cap I_y \neq \emptyset$. Remember also that $z \in N$ by assumption. Thus, by Claim 1, no interval of a vertex in $\{x, y, z\}$ properly contains an interval of another vertex in $\{x, y, z\}$.

Let $I_x = [x_1, x_2]$, $I_y = [y_1, y_2]$, $I_z = [z_1, z_2]$, and since $I_x \cap I_y \neq \emptyset$ and one does not properly contain the other, without loss of generality assume that $x_1 < y_1 < x_2 < y_2$. We now have two cases regarding the position of $I_z$ with respect to $I_y$.

- First assume that $I_z$ does not overlap $I_y$. If $y_2 < z_1$, consider an $x, z$-path $P_{x,z}$ in $G$ that avoids $N(y)$. Here $I_y \subset [x_1, z_2]$, and thus by Claim 2, there exists a non-probe internal vertex on $P_{x,z}$, as required. If $z_2 < y_1$, then $I_z \subset [z_1, y_2]$, and thus by Claim 2, there exists a non-probe internal vertex $v$ of a $y, z$-path such that $I_x \subset I_v$, as required.
- If $I_z$ overlaps $I_y$ (remember that $x, y, z \in N$, so by Claim 1, $I_z \not\subset I_y$ and $I_y \not\subset I_z$), then we have three possible cases:
  - If $y_1 < z_1 < y_2 < z_2$, then $I_z \subset [x_1, z_2]$ and thus, by Claim 2, there must exist a non-probe internal vertex $v$ on an $x, z$-path such that $I_v \supseteq I_y$, as required.
  - If $z_1 < y_1 < z_2 < x_2$ (this implies that $z_1 < x_1$, since $I_z \not\subset I_x$), then $I_x \subset [z_1, y_2]$ and thus, by Claim 2, there must exist a non-probe internal vertex $v$ on an $y, z$-path such that $I_v \supseteq I_x$, as required.
  - If $I_x \cap I_y \subset I_z$, then $x_1 < z_1 < y_1 < x_2 < z_2 < y_2$; this is because $I_z \not\subset I_x$ and $I_y \not\subset I_z$ by Claim 1. Now $I_z \subset [x_1, y_2]$ and thus, by Claim 2, there must exist a non-probe internal vertex $v$ on an $x, y$-path such that $I_v \supseteq I_z$, as required.

(2) Now consider the case when $xy \notin E_H$ and $uz \in E_H$ for some $u \in V(P_{x,y})$.

- If $u \not\in \{x, y\}$, then since $uz \notin E_G$, both $u$ and $z$ are non-probes, and $I_u \cap I_z \neq \emptyset$. Thus, an internal vertex $u$ of $P_{x,y}$ is a non-probe, as required.
- If $u \in \{x, y\}$, without loss of generality let $u = x$, then since $xz \notin E_G$, $I_x \cap I_z \neq \emptyset$ and $x, z \in N$. Since $xy \notin E_H$, $I_x \cap I_y = \emptyset$; $y$ could be a probe or a non-probe. Without loss of generality let $x_1 < x_2 < y_1 < y_2$. By Claim 1, $I_z \not\subset I_x$ and $I_x \not\subset I_z$. Thus we have only two cases to consider: $z_1 < x_1 < z_2 < x_2$ and $x_1 < z_1 < x_2 < z_2$. If $z_1 < x_1 < z_2 < x_2$, then $I_x \subset [z_1, y_2]$ and thus, by Claim 2, there exists an internal non-probe vertex $v$ on a $y, z$-path such that $I_x \subset I_v$, as required. Similarly, if $x_1 < z_1 < x_2 < z_2$, then $I_z \subset [x_1, y_2]$ and thus, by Claim 2, there exists an internal non-probe vertex $v$ on a $x, y$-path such that $I_v \supseteq I_z$, as required. □
The following is a straightforward corollary of Claim 3.

**Corollary 2.** There exists only one $(P, N)$-partition of vertices of a 3-sun up to isomorphism.

**Proof.** Consider a 3-sun $G$ labeled as in Fig. 2 with the AT-vertices $x, y, z$. By Lemma 1, at least one of $x, y, z$ is a non-probe. Without loss of generality, let $z \in N$. Thus, $N(z) = \{a, b\} \subseteq P$. If $x \in N$, then $u \in P$, so all internal vertices of all paths between AT-vertices are probes contradicting Claim 3. Thus, $x \in P$, and similarly, $y \in P$. By Claim 3, $u \in N$. □

### 4. 2-Tree probe interval graphs

We define a 2-path recursively in the following way:

- A triangle, $K_3$, is a 2-path of length one; denote the triangle by $t_1$.
- $t_0 = \emptyset$.
- If $A$ is a 2-path of length $k (k \geq 1)$ with the triangle sequence $t_1t_2\ldots t_k$, a new length $(k+1)$ 2-path is obtained by adding to $A$ a vertex $v$ and edges $vv_1$ and $vv_2$, where $v_1v_2$ is an edge of $t_k \setminus t_{k-1}$; the new triangle induced on $\{v, v_1, v_2\}$ is denoted by $t_{k+1}$.

An example of a 2-path is presented in Fig. 3. We say that the triangles $t_i$ and $t_{i+1}$, $1 \leq i \leq k - 1$, of a 2-path $A$ are consecutive triangles of $A$, and that two triangles are adjacent if they share an edge. Triangles $t_1$ and $t_k$ of a length $k$ 2-path are called end triangles. A vertex $v$ of degree 2 of an end triangle $t_i$ or $t_k$ of a 2-path $A = t_1 \ldots t_k$ is called an end vertex of $A$; if $k \geq 2$, we denote by $v_1$ the degree 2 vertex of $t_1$, and by $v_k$ the degree 2 vertex of $t_k$. An edge $e$ of an end triangle containing an end vertex is called an end edge. An edge of a 2-path $A$ that is not shared between 2 triangles of $A$ and is not an end edge of $A$ is called a side edge of $A$. A non-end, non-side edge of a 2-path $A$ is called an internal edge of $A$. Clearly, the length of a 2-path $A$, denoted by $l(A)$, is the number of triangles in it. Denote by $A_i$ a 2-path of length $i$. The distance between two triangles is the number of edges shared between pairs of consecutive triangles on the shortest 2-path between them.
Fig. 3. A 2-path of length 11 with examples of: end triangles $t_1$ and $t_{11}$, end vertices $v_1$ and $v_{11}$, end edges $e_1, e_2, e_3,$ and $e_4$, side edges $s_1$ and $s_2$, and an internal edge $i_7$.

Fig. 4. $A_2, A_3, A_4, A_5, A_6$ non-isomorphic $A_4$s, non-isomorphic $A_5$s, and non-isomorphic $A_6$s.

**Observation 1.** There exists one $A_2$, one $A_3$, two non-isomorphic $A_4$s, three non-isomorphic $A_5$s, and six non-isomorphic $A_6$s.

**Proof.** By inspection, there are two ways of identifying an edge of an $A_3$ with an edge of an $A_1$ to obtain an $A_4$, three ways of identifying an edge of an $A_4$ with an edge of an $A_1$ to obtain an $A_5$, and six ways of identifying an edge of an $A_5$ with an edge of an $A_1$ to obtain an $A_6$. They are all presented in Fig. 4. □

By identifying a side edge of an $A_5$ with an end edge of the $A_2$ in all possible ways so that the resulting 2-tree still has a longest 2-path of length 5, we obtain the two non-isomorphic 2-trees presented in Fig. 5. We call graphs $S_1$ and $S_2$ presented in Fig. 5 weak 2-stars.

**Claim 4.** No weak 2-star is a PIG.

**Proof.** Assume a weak 2-star is a PIG. Consider ACs $\{X, Y, Z\}$ of $S_1$ and $S_2$ from Fig. 5, where $X = \{x, x_1\}, Y = \{y, y_1\},$ and $Z = \{z, z_1\}$. None of the ASs $X$, $Y$, and $Z$ of $S_1$ and $S_2$ is an independent set contradicting Corollary 1. □

Consider a 2-path $A$ of length at least 3 of a 2-tree $T$ and denote by $v_1 v_2$ a side edge of $A$. For a vertex $v \notin V(A)$ of $T$ such that $v v_1, v v_2 \in E(T)$ we say that the triangle $v v_1 v_2$ is an additional triangle at distance 1 from $A$ and that $v$ is an additional vertex at distance 1 from
A; the number of edges on a shortest 2-path between the triangle \( v_1 v_2 \) and a triangle of \( A \) is 1. Now consider a 2-path \( A \) with an additional triangle \( v_1 v_2 \) at distance 1 from \( A \) in a 2-tree \( T \). For a vertex \( u \not\in V(A) \cup \{v\} \) of \( T \) such that \( uv, uvi \in E(T) \) for exactly one \( i \in \{1, 2\} \), we say that the triangle \( uvv_i \) is an additional triangle at distance 2 from \( A \) and that \( u \) is an additional vertex at distance 2 from \( A \); the number of edges on a shortest 2-path between the triangle \( uvv_i \) and a triangle of \( A \) is 2. Similarly, we can define additional triangles at distance 3 or more from \( A \). We will use the phrase an additional triangle with respect to \( A \) to refer to an additional triangle at distance \( i \geq 1 \) from \( A \). When it is clear from the context which \( A \) is being considered, we will omit reference to \( A \).

**Claim 5.** Let \( T \) be a 2-tree PIG and let \( A \) be a longest 2-path of \( T \). \( T \) contains no additional triangles at distance 2 from \( A \).

**Proof.** Assume to the contrary. Let \( A = t_1 t_2 \ldots t_m \), where \( t_1, \ldots, t_m \) are consecutive triangles of \( A \), and let \( p \) be an additional triangle at distance 2 from \( A \). Let \( p \) be at distance 2 from some \( t_i \) of \( A \), and let \( q \) be the triangle having an edge in common with \( p \) and an edge in common with \( t_i \). Since \( A \) is longest, we know that \( 3 \leq i \leq m - 2 \). But now the subgraph of \( T \) induced on the union of the vertices of triangles \( t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}, q \), and \( p \) is a weak 2-star contradicting \( T \) being a PIG, by Claim 4.

From Claim 5 it follows that if additional triangles with respect to a longest 2-path \( A \) of a 2-tree PIG exist, then they must be at distance 1 from \( A \). The next claim determines to which of the \( P \) and \( N \) vertex partitions of a 2-tree PIG the degree 2 vertices of the additional triangles belong. Recall that we denote by \( v_1 \) the degree 2 vertex of \( t_1 \), and by \( v_m \) the degree 2 vertex of \( t_m \) in a 2-path \( A_m \).

**Claim 6.** Let \( T \) be a 2-tree PIG, let \( A = t_1 t_2 \ldots t_m \) be a longest 2-path of \( T \), and let \( l(A) \geq 4 \). Let \( t \) be an additional triangle at distance 1 from \( A \). Denote by \( v \) the degree 2 vertex of \( t \).

- If \( t \) is adjacent to \( t_2 \) (or equivalently, to \( t_{m-1} \)) and if \( v \in P \), then \( v_1 \in N \) (\( v_m \in N \)).
- If \( t \) is adjacent to \( t_i \) for \( 3 \leq i \leq m - 2 \), then \( v \in N \).
Proof. Assume to the contrary. First, let \( t \) be adjacent to \( t_2 \) and \( v, v_1 \in P \). Let the vertices in \( V(t_1) \cup V(t_2) \cup V(t_3) \cup V(t_4) \cup \{ v \} \) be labeled as in Fig. 6A, which illustrates the only two non-isomorphic \( A_4 \)s (by Observation 1) with an additional triangle at distance 1 that is adjacent to \( t_2 \) (or equivalently to \( t_{m-1} \)). Since the subgraph of \( T \) induced on \( \{ v, v_1, 1, 2, 3, 4 \} \) is a 3-sun with the AT \( \{ v, v_1, 4 \} \) and since \( v, v_1 \in P \), by Corollary 2, we know that \( 1, 4 \in N \).

Since \( 4 \in N \) and \( 45 \in E(T) \), we must have \( 5 \in P \). But now \( \{ v, v_1, 5 \} \) is an all-probe AT in \( T \) contradicting Lemma 1. The proof is the same for the case when \( t \) is adjacent to \( t_{m-1} \).

Now assume that \( t \) is adjacent to \( t_i \) for \( 3 \leq i \leq m-2 \) and \( v \in P \). Clearly, \( m \geq 5 \). Label the vertices of \( V(t_{i-2}) \cup V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t_{i+2}) \cup \{ v \} \) as in Fig. 6B, which illustrates the only three non-isomorphic \( A_5 \)s (by Observation 1) with an additional triangle at distance 1 that is adjacent to their \( t_3 \)s. Clearly, the subgraph of \( T \) induced on \( \{ 1, 2, 3, 4, 5, v \} \) is a 3-sun, and since \( \{ v, 3, 5 \} \) is an AT, if we assume that \( v \in P \), then by Lemma 1, either \( 3 \in N \), or \( 5 \in N \). Without loss of generality, let \( 3 \in N \). Then by Corollary 2, vertex 2 is also in \( N \). Since \( 25, 36 \in E(T) \), \( 5, 6 \in P \). But now we have an all probe AT \( \{ v, 5, 6 \} \) in \( T \) contradicting Lemma 1. \( \square \)

We call a graph obtained by a \( K_2 \)-bonding of an end edge of an \( A_3 \) with the internal edge \( e = t_3 \cap t_4 \) of an \( A_6 = t_1 \ldots t_6 \) a 2-star. There exist two ways of identifying an end edge of an \( A_3 \) with the edge \( e = t_3 \cap t_4 \) of an \( A_6^1 \) from Fig. 4 to obtain 2-stars \( S_3 \) and \( S_4 \) presented in Fig. 7, two ways of identifying an end edge of an \( A_3 \) with the edge \( e = t_3 \cap t_4 \) of an \( A_6^2 \) from Fig. 4 to obtain 2-stars \( S_5 \) and \( S_6 \) presented in Fig. 7, and four ways of identifying an end edge of an \( A_3 \) with the edge \( e = t_3 \cap t_4 \) of each of \( A_6^3, A_6^4, A_6^5, \) and \( A_6^6 \) from Fig. 4 to obtain sixteen more 2-stars. In this way we constructed twenty 2-stars in total. However, many of them are isomorphic (we tested these isomorphisms manually, as well as by using McKay’s Nauty 2.0 software [11]). Thus, the following claim holds.

![Fig. 6. (A) The two A4s with an additional triangle. (B) The three A5s with an additional triangle.](image1)

![Fig. 7. The ten non-isomorphic 2-stars.](image2)
Claim 7. There exist ten non-isomorphic 2-stars. They are denoted by $S_3, \ldots, S_{12}$ and presented in Fig. 7.

Note that $S_9$ presented in Fig. 7 contains an induced $G_4$ presented in Fig. 1, which is a non-PIG tree, by Theorem 2. Thus, from now on, we exclude $S_9$ from the set of 2-stars, since our goal here is to describe non-PIG 2-trees that do not contain non-PIG trees as induced subgraphs.

Claim 8. No 2-star is a PIG.

Proof. This is because no AS of the AC $\{X, Y, Z\}$ of any of the 2-stars is independent, and thus every 2-star has an all probe AT contradicting Lemma 1. □

Similar to the definition of additional triangles, we now define triangles that “grow off” an internal edge of a longest 2-path of a 2-tree PIG, rather than off a side edge as was the case for additional triangles. Consider a 2-path $A$ of length at least 2 of a 2-tree $T$ and denote by $v_1v_2$ an internal edge of $A$. For a vertex $v \notin V(A)$ of $T$ such that $v_1v, v_2v \in E(T)$ we say that the triangle $v_1v_2$ is an extra triangle at distance 1 from $A$ and that $v$ is an extra vertex at distance 1 from $A$; the number of edges on a shortest 2-path between the triangle $v_1v_2$ and a triangle of $A$ is one. Now consider a 2-path $A$ with an extra triangle $v_1v_2$ at distance 1 from $A$ in a 2-tree $T$. For a vertex $u \notin V(A) \cup \{v\}$ of $T$ such that $uv, uv_1 \in E(T)$ for exactly one $i \in \{1, 2\}$, we say that the triangle $uv_1v_2$ is an extra triangle at distance 2 from $A$ and that $u$ is an extra vertex at distance 2 from $A$; the number of edges on a shortest 2-path between the triangle $uv_1v_2$ and a triangle of $A$ is two. Similarly, we define an extra triangle at distance 3 from $A$ as the triangle $uwx$ where $w \notin V(A) \cup \{u, v\}$ and $x \in \{v, v_1\}$, and an extra vertex $w$ at distance 3 from $A$. We will use the phrase an extra triangle with respect to $A$ to refer to an extra triangle at distance $i \geq 1$ from $A$. When it is clear from the context what $A$ is being considered, we will omit reference to $A$.

Claim 9. A 2-tree PIG $T$ does not contain any extra triangles at distance 3 or more from a longest 2-path $A$ of $T$.

Proof. Otherwise, $T$ would contain an induced 2-star contradicting it being a PIG, by Claim 8. In particular, if $t$ is an extra triangle that is at distance 3 from $A$, and if two shortest 2-paths between $t$ and a triangle of $A = t_1 \ldots t_m$ are $tpqt_i$ and $tpqt_{i+1}$, where $t_i$ and $t_{i+1}$ are two consecutive triangles of $A$, then we know that $3 \leq i < i+1 \leq m-2$, since otherwise $A$ would not have been a longest 2-path of $T$ (clearly, $m \geq 6$). But now a subgraph of $T$ induced on the vertices in $V(t_{i-2}) \cup V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t_{i+2}) \cup V(t_{i+3}) \cup V(p) \cup V(q) \cup V(t)$ is a 2-star, contradicting $T$ being a PIG, by Claim 8. □

From Claim 9 it follows that if extra triangles with respect to a longest 2-path $A$ of a 2-tree PIG exist, then they must be at distance 1 or 2 from $A$. The next claim determines the partition to which the degree 2 vertices of the distance 2 extra triangles of $A$ belong. As before, we denote by $v_1$ the degree 2 vertex of $t_1$, and by $v_m$ the degree 2 vertex of $t_m$.
Claim 10. Let $T$ be a 2-tree PIG, let $A = t_1 t_2 \ldots t_m$ be a longest 2-path of $T$, and let $m \geq 4$. Let $t$ be an extra triangle at distance 2 from $A$. Denote by $v$ the degree 2 vertex of $t$. If $t$ is at distance 2 from $t_2$ and $t_3$ (or equivalently, from $t_{m-1}$ and $t_{m-2}$) then:

- if $v \in P$, then $v_1 \in N (v_m \in N)$;
- if $m = 4$ and $v \in P$, then either $v_1$, or $v_m$, or both are non-probe.

If $t$ is at distance 2 from $t_i$ and $t_{i+1}$, for $3 \leq i \leq m - 3 (m \geq 6)$, then $v \in N$.

Proof. If $m = 4$ and if all three vertices $v, v_1$ and $v_4$ are probe, than they form an all probe AT in $T$ contradicting Lemma 1. Let $m \geq 5$ and let $t$ be at distance 2 from $t_2$ and $t_3$ (or equivalently from $t_{m-1}$ and $t_{m-2}$). Denote by $u, w$ the vertices of $V(t_5) \setminus V(t_3)$ (or vertices of $V(t_{m-4}) \setminus V(t_{m-2})$ if $t_{m-1}$ and $t_{m-2}$ are being considered). Now $\{X, Y, Z\}$ where $X = \{v_1\}$, $Y = \{v\}$, $Z = \{u, w\}$ is an AC, so if $v \in P$, since $Z$ is not independent, $v_1$ must be a non-probe, by Lemma 1.

We now consider the case when $t$ is at distance 2 from $t_i$ and $t_{i+1}$ for $3 \leq i \leq m - 3 (m \geq 6)$. Let $u_1, w_1$ be the vertices of $V(t_{i-2}) \setminus V(t_i)$, and let $u_2, w_2$ be the vertices of $V(t_{i+3}) \setminus V(t_{i+1})$. Now $\{X, Y, Z\}$ is an AC, where $X = \{u_1, w_1\}$, $Y = \{u_2, w_2\}$, and $Z = \{v\}$, so since $X$ and $Y$ are not independent, $v$ must be a non-probe, by Lemma 1. □

We now describe some structure of 2-tree PIGs that is forced by the existence of additional and extra triangles with respect to their longest 2-paths.

Claim 11. Let $t'$ be an additional triangle at distance 1 from a longest 2-path $A = t_1 \ldots t_m$ of a 2-tree PIG $T$, and let $t''$ be an extra triangle at distance 2 from $A$. Let $t'$ be adjacent to the triangle $t_i$ and let $t''$ be at distance 2 from $t_i$ and $t_{i+1}$ of $A$ for $3 \leq i \leq m - 3 (m \geq 6)$, let $v'$ be the degree 2 vertex of $t'$, let $v''$ be the degree 2 vertex of $t''$, and let $s$ be the vertex in $t_i$ that is not in $t'$. Then $v'' \notin E(T)$.

Proof. By Claim 6, $v' \in N$. Thus, $s \in N$, by Corollary 2 applied to the 3-sun induced on vertices of $t_{i-1} \cup t_i \cup t_{i+1} \cup t'$. Also, $v'' \in N$, by Claim 10. Now, since $s, v'' \in N$, they cannot be adjacent. □

Corollary 3. The twenty-seven 2-trees presented in Fig. 8 are minimal non-PIGs.

Proof. First we show that none of the graphs in Fig. 8 are PIGs. Notice that vertices $a$ and $7$ in graphs $S_{13}, S_{14},$ and $S_{15}$, as well as vertices $a$ and $8$ in graphs $S_{16}, S_{17},$ and $S_{18}$ violate the
conditions of Claim 11 and thus these graphs are not PIGs. We reason about graphs $S_{14}^i$, $S_{15}^j$, and $S_{17}^l$ ($i, j, l \geq 2$) as follows. Assume they are PIGs. By Claim 6, $a \in N$, which implies that $s \in N$ by applying Corollary 2 to the 3-sun induced on vertices $\{1, 2, 3, 2, a, s\}$ in $S_{14}^i$, $S_{15}^j$ and $S_{17}^l$. This further implies that all neighbors of $s$ are in $P$ in these graphs, and thus, if $i, j, l \geq 3$, the graph induced on vertices $3_n, 3_{n-1}, 3_{n-2}, 4, 5, 6, 7$ in $S_{14}^i (n = i)$ and $S_{15}^j (n = j)$, and vertices $\{3_1, 3_1-1, 3_1-2, 5, 6, 7, 8\}$ in $S_{17}^l$ is isomorphic to graph $G_5$ in Fig. 1 with vertex $3_n (n = i, j, l$ for graphs $S_{14}^i$, $S_{15}^j$, and $S_{17}^l$, respectively) corresponding to vertex $w$ of $G_5$ and being a probe, contradicting Theorem 1; similarly, if $i, j, l = 2$, then the graph induced on vertices $3_2, 3_1, 2, 4, 5, 6, 7$ in $S_{14}^2$ and $S_{15}^2$, as well as vertices $3_2, 3_1, 2, 5, 6, 7, 8$ in $S_{17}^2$ is isomorphic to $G_5$ with $3_2 \in P$ contradicting Theorem 1. In graphs $S_{16}^k$ and $S_{18}^m$ presented in Fig. 8, $a, s \in N$ by Claim 6 and Corollary 2, and thus $x, y, z$ is an all-probe AT contradicting Lemma 1, where $x \in X$, $y \in Y$, and $z \in Z$. It is straightforward to verify that the graphs presented in Fig. 8 are non-isomorphic, and that they are minimal non-PIGs (deletion of any vertex from these graphs yields a PIG).

Note that all graphs presented in Fig. 8 apart from $S_{13}$ contain a fan of some small size. We cannot insert a fan into $S_{13}$ in the same way we did in the other graphs in Fig. 8 for the following reasons. If we insert a $k$-fan next to vertex $3_1$, then $S_{13}^k$ would contain two trees isomorphic to $G_5$ presented in Fig. 1, one induced on vertices $5, 8, 7, 6, 3_k, 3_k-1, 3_k-2$ (or in the case that $k = 2$, induced on vertices $5, 8, 7, 6, 3_2, 3_1, a$), and the other induced on vertices $0, 1, s, 3_1, a, 4, 5$. The vertex $3_k$ in the first copy of $G_5$ in $S_{13}^k$ corresponds to the vertex $w$ of $G_5$ in Fig. 1, and so does the vertex $s$ in the second copy of $G_5$ in $S_{13}^k$. However, $3_k$ and $s$ are adjacent in $S_{13}^k$, which contradicts the condition of Theorem 1 that they both have to be non-probe. No fans larger than the ones indicated in Fig. 8 can be inserted into the other graphs in Fig. 8, since otherwise removal of the vertex $3_{[k/2]}$ would yield a non-PIG contradicting the minimality of these graphs.

Claim 12. Let $t$ be an extra triangle at distance 1 from a longest 2-path $A = t_1 \ldots t_m$ of a 2-tree PIG $T$, and let $t$ be adjacent to $t_1$ and $t_{i+1}$ of $A$, for $3 \leq i \leq m - 3 (m \geq 6)$. There do not exist in $T$ two extra triangles $t_1'$ and $t_2'$ at distance 2 from $A$ such that they are adjacent to different edges of $t$. If $i = 2$ or $i = m - 2$, then:

- exactly one of $t_1'$ and $t_2'$ must have its degree 2 vertex a probe;
- if in addition $m = 5$ and the vertices of $T$ are denoted as in Fig. 9 ($T$ can be any one of $T_1, T_2$, and $T_3$ in Fig. 9), or if $m = 4$ and $T = T_2 \setminus \{7\}$, where $T_2$ is presented in Fig. 9, then $v_1' \in N$ and $v_2' \in P$.

Proof. Assume to the contrary. Denote by $v_1'$ the degree 2 vertex of $t_1'$ and by $v_2'$ the degree 2 vertex of $t_2'$. First, let $3 \leq i \leq m - 3 (m \geq 6)$. Now the subgraph $S$ of $T$ induced on the union of the vertices of triangles $t_i, t, t_1'$, and $t_2'$ is a 3-sun, but by Claim 10 two of its AT-vertices $v_1'$ and $v_2'$ are non-probe, contradicting Corollary 2.

Now let $i = 2$ (or equivalently $i = m - 2$). Denote a subset of the vertices of $T$ as in Fig. 9 ($T$ can be any one of $T_1, T_2$, and $T_3$). If both $v_1'$ and $v_2'$ are non-probes, then the
Fig. 9. The three (valid) 2-tree PIGs with two extra triangles at distance 2 from a longest 2-path $A = t_1 \ldots t_5$ such that they are adjacent to the same extra triangle.

Fig. 10. Six minimal forbidden induced subgraphs for 2-tree PIGs resulting from Claim 12.

3-sun $S$ induced on the set of vertices $\{2, 3, 4, v'_1, v, v'_2\}$ has two non-probe AT-vertices contradicting Corollary 2. If both $v'_1$ and $v'_2$ are probes, then by Corollary 2 applied to $S$, $2 \in N$ and thus $1 \in P$; this contradicts Claim 10 which says that vertex 1 must be a non-probe, since $v'_1, v'_2 \in P$. Thus, one of $v'_1$ and $v'_2$ must be a probe and the other one non-probe. Note that if $m = 5$, we must have $v'_1 \in N$ and $v'_2 \in P$, since otherwise we would have $3 \in N$ and thus $1 \in P$, contradicting Claim 10 (another way to see this is: if we would have $v'_2 \in N, v'_1 \in P$, then either $\{1, v'_1, 6\}$, or $\{1, v_1, 7\}$, or both would form an all-probe AT contradicting Lemma 1). If $m = 4$, in the 2-tree PIG $T_1 \setminus \{7\} = T_3 \setminus \{7\}$ presented in Fig. 9 we can have $v'_2 \in N, v'_1 \in P$, in which case $6 \in N$, by Claim 10. However, if $m = 4$ and in $T_2 \setminus \{7\}$ we have $v'_2 \in N$ and $v'_1 \in P$, then $T_2 \setminus \{7\}$ would contain an all-probe AT $\{v'_1, 1, 6\}$ contradicting Lemma 1; thus we must have $v'_2 \in P$ and $v'_1 \in N$ in $T_2 \setminus \{7\}$. □

**Corollary 4.** The six 2-trees presented in Fig. 10 are minimal non-PIGs.

**Proof.** There are six non-isomorphic $A_6$s, by Observation 1. Using the same notation as in Claim 12, since the “addition” of $t, t'_1, t'_2$ to each of the six non-isomorphic $A_6$s does not increase the length of the longest path in the resulting graph, we conclude that the six 2-trees presented in Fig. 10 are non-isomorphic. They are non-PIGs, since they violate the conditions of Claim 12. It is easy to see that the removal of any vertex from $S_i$, $i \in \{19, \ldots, 24\}$ yields a PIG, that is, $S_{19}, \ldots, S_{24}$ are minimal non-PIGs. □

**Claim 13.** Let $v'_1$ and $v'_2$ be the degree 2 vertices of two different additional triangles $t'_1$ and $t'_2$ at distance 1 from a longest 2-path $A = t_1 \ldots t_m$ of a 2-tree PIG $T$. Let $t'_i$ be adjacent to $t_i$ and let $t'_j$ be adjacent to $t_j$ of $A$, $3 \leq i < j \leq m - 2$, and denote by $s'_1$ and $s'_2$ the vertices in $V(t_i) \setminus V(t'_1)$ and $V(t_j) \setminus V(t'_2)$, respectively. Then $v'_1 s'_2$, $v'_2 s'_1 \notin E(T)$. 
If $v_1', v_2' \in E(T)$, then at least one of $v_1', v_2'$ must be a probe which contradicts the fact that both of them must be non-probe: $v_1' \in N$ by Claim 6, and $v_2' \in N$ by Corollary 2 since it belongs to the 3-sun formed by the union of triangles $t_{j-1}, t_j, t_{j+1}, t_2'$, and since $v_2' \in N$, by Claim 6. □

Corollary 5. The eleven 2-trees presented in Fig. 11A are minimal non-PIGs.

Proof. It follows directly from Claim 13 that graphs $S_{25}, S_{26}, S_{27}, i \in \{1, \ldots, 5\}$ presented in Fig. 11A are not PIGs, since vertices $a_1, s_1, a_2$, and $s_2$ in $S_{25}$ as well as vertices $a_1, s, a_2$, and $3_i$ in graphs $S_{26}$ and $S_{27}$ violate the condition described in Claim 13. It is easy to see that these graphs are minimal non-PIGs, since removal of any vertex from any of them yields a PIG. □

Similar to the explanation given after the proof of Corollary 3, no fans can be inserted in the graph $S_{25}$ in Fig. 11A, and no fan larger than a 5-fan can be inserted in the other two graphs in the same figure.

Claim 14. Let $t_1'$ and $t_2'$ be additional triangles at distance one from a longest 2-path $A = t_1 \ldots t_m$ of a 2-tree $T$ that are adjacent to triangles $t_i$ and $t_{i+2}$ of $A$, respectively, $3 \leq i < m - 1$, such that there exists a vertex $u$ which satisfies $\{u\} = V(t_i) \cap V(t_{i+2})$ and $\{u\} = V(t_1') \cap V(t_2')$. Then $T$ is not a PIG.

Proof. Denote by $a_1$ the degree 2 vertex of $t_1'$, by $a_2$ the degree 2 vertex of $t_2'$, by $s_1$ the vertex in $V(t_j) \setminus V(t_1')$, and by $s_2$ the vertex in $V(t_{i+2}) \setminus V(t_2')$. Clearly, $s_1 s_2 \in E(T)$, by definition of $A$ and $u$. Assume that $T$ is a PIG. Since $t_1'$ is adjacent to $t_i$, $3 \leq i \leq m - 3$, by Claim 6, $a_1 \in N$. Consider the position of the vertex $a_2$ with respect to $A$.

- If $i + 2 \leq m - 2$, by Claim 6, we conclude that $a_2 \in N$. In this case, both $s_1$ and $s_2$ are non-probe, because the subgraph of $T$ induced on $V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t_1')$ is a 3-sun with an AT vertex $a_1$ being a non-probe, and thus by Corollary 2, $s_1 \in N$; similarly, the subgraph of $T$ induced on $V(t_{i+1}) \cup V(t_{i+2}) \cup V(t_{i+3}) \cup V(t_2')$ is a 3-sun with an AT vertex $a_2$ being a non-probe, and thus by Corollary 2, $s_2 \in N$. This contradicts $s_1$ and $s_2$ being adjacent.
- If $i + 2 = m - 1$ and $a_2 \in N$, the same argument as above leads to a contradiction.
- If $a_2 \in P$, then the following argument leads to a contradiction. We know that $u$, as a
neighbor of $a_1 \in N$, is in $P$. Thus, since $a_2$ is also a probe, in the 3-sun induced on $V(t_{m-2}) \cup V(t_{m-1}) \cup V(t_m) \cup V(t_i')$, by Corollary 2, $v_m \in P$, where $v_m$ is the degree 2 vertex of $t_m$. Since in the 3-sun induced on the vertices of $V(t_{i-1}) \cup V(t_i) \cup V(t_{i+1}) \cup V(t_i')$ we know that $a_1, s_1 \in N$ (the proof is above), this implies that all neighbors of $s_1$ must be probe. Consider the neighbor $v_{i-1} \in V(t_{i-1}) \setminus V(t_i)$ of $s_1$. Now vertices $\{a_2, v_m, v_{i-1}\}$ form an all-probe AT in $T$ contradicting Lemma 1. □

Corollary 6. The five 2-trees presented in Fig. 11B are minimal forbidden induced subgraphs for 2-tree PIGs.

Proof. The proof that these graphs $S^1_{28}$ and $S_{29}$ are not PIG follows directly from Claim 14. For graphs $S^i_{28}$, $i \in \{2, \ldots, 4\}$, the proof is similar to the proof of Claim 14: it is easy to see that $a_1, s_1 \in N$ by Claim 6 and Corollary 2, and thus all neighbors of $s_1$ are probe; also $v_{i-1}$ is a non-probe, by Corollary 2 applied to the 3-sun induced on $\{s_1, s_2, v_m, v_{i-1}, a_2, u_i\}$, since $s_1 \in N$, and thus $v_m, a_2 \in P$ as neighbors of $s_1$; now $\{v_m, a_2, v_{i-1}\}$ form an all-probe AT in $S^i_{28}, i \in \{2, \ldots, 4\}$ contradicting Lemma 1.

It is easy to see that the removal of any vertex from any of these graphs makes the resulting graph PIG, that is, these graphs are minimal non-PIG 2-trees. □

Similar to the explanation given after the proof of Corollary 3, no fans can be inserted in the graph $S_{29}$ in Fig. 11B, and no fan larger than a 4-fan can be inserted in the other graph in the same figure.

Combining Theorem 2, Claims 4, 8, and Corollaries 3–6, we have the following:

Theorem 3. There exist at least 62 graphs in the forbidden induced subgraph characterization for 2-tree PIGs.

5. Conclusions and future work

We have shown that the FISC for 2-tree PIGs contains at least 62 graphs. It is possible that this list is complete. However, the key point is that this FISC is not concise and thus does not seem to give much insight into the structure of 2-tree PIGs.

It is interesting to note that 13 out of 14 forbidden induced subgraphs for PIGs described in Theorem 2, Claim 4, and Claim 8 have asteroidal triples of edges, a structure introduced by Müller [13]: three edges $e_1, e_2, e_3$ of a graph $G$ form an asteroidal triple of edges (ATE) if for any two of them there is a path from the vertex set of one to the vertex set of the other that avoids the neighborhood of the third edge, where a neighborhood of an edge $e = uv$ is $N(u) \cup N(v)$. However, the remaining 49 forbidden induced subgraphs for PIGs do not have ATEs and it is not clear if other more general structures occur in these subgraphs. Note that our Corollary 1 is similar to the previously shown result that PIGs cannot have ATEs [3]. Other related results that have appeared recently include a FISC for tree unit PIGs [5] and unit interval bigraphs [4]; in unit PIGs all intervals in an interval representation of a PIG are of the same length, while unit interval bigraphs are bipartite intersection graphs of two distinct families of the same length intervals with two vertices.
adjacent if and only if their corresponding intervals overlap and each interval belongs to a
distinct family.

Sheng’s FISC for tree PIGs [15] implies the existence of an efficient algorithm for solving
the non-GP recognition problem for tree PIGs. Using Shamir and Tsur’s subtree isomor-
phism algorithm [14] to determine if each of the two trees in the FISC for tree PIGs (the
graphs $G_4$ and $G_6$ presented in Fig. 1) is present in a tree $T$ yields an $O(n)$ algorithm for
determining if $T$ is a PIG, where $n$ is the number of vertices in $T$. The problem of efficient
non-GP recognition of 2-tree PIGs remains open even if we know a complete FISC for 2-tree
PIGs. The more general problems of non-GP recognition of $k$-tree PIGs for any positive
integer $k$, chordal PIGs, and PIGs in general remain open as well.

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References

[1] A. Berry, M.C. Golumbic, M. Lipshteyn, Cycle-bicolorable graphs and triangulating chordal probe graphs,
[2] A. Berry, M.C. Golumbic, M. Lipshteyn, Two tricks to triangulate chordal probe graphs in polynomial time,
962–969.
[7] M.C. Golumbic, M. Lipshteyn, Chordal probe graphs, Twenty-ninth Workshop on Graph Theoretic Concepts
[17] P. Zhang, Methods of mapping DNA fragments, United States Patent, Online available at