3C04 - Exercises 3 and 4

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Exercises marked with a star are a little more difficult than others. Exercises marked with two stars are more challenging still.

Questions 7 to 10 are the most important on this handout as they involve practising the central skill in the course. You should definitely attempt all four.

1. On the basis of the definitions used in the lecture notes, decide whether each of the following statements is true or false, and explain your reasoning.

    - If a Turing machine (TM) semidecides a language L then L is recursive.
    - If a TM decides a language L then L is recursive.
    - If a TM semidecides language L then L is r.e.
    - If a TM decides language L then L is r.e.
    - If a TM does not accept an input x then it fails to halt.
    - If a TM recognizes a language L then inputs *not* in L will be rejected.
    - If a TM produces halts in the Y state when given an input x then we can say that the TM has 'recognized' x.

2. $\star$Consider the proof given in the lectures that the set of all languages for a finite alphabet is uncountable. The proof relies on the use of a diagonal argument to construct a new language. Is the language that is constructed recursive?

3. Explain in detail why the condition that $x \in A \leftrightarrow f(x) \in B$ appearing in the definition of reduction is necessary.

4. Explain in detail why the function f used in the definition of reduction has to be computable.

5. Explain in detail why it is incorrect to try to prove a problem unsolvable by reducing it to a known unsolvable problem.

6. $\star$Let L be a language that is known to be recursively enumerable (r.e.) although we don't know whether or not it's recursive. Let L_2 be a second language, where we have succeeded in proving that L reduces to L_2. Explain our state of knowledge about L_2. Is it necessarily the case that L_2 can be reduced to L? Explain your answer.

7. $\star$Consider the following decision problem.

    Instance Two Turing machines, M_1 and M_2, and an input alphabet Σ_I common to both machines.

    Question Is it the case that there is *exactly one* input $x \in \Sigma_I^\star$ for which $f_{M_1}(x) = f_{M_2}(x)$?

Expanding a little, note that f_{M_1} and f_{M_2} are the functions computed by[1] the two Turing machines. The answer to the question is:

- 'yes', if there is *exactly one* $x \in \Sigma_I^\star$ for which $f_{M_1}(x) = f_{M_2}(x)$.
- 'no', in any other circumstances. For example, if there are two inputs x_1 and x_2 such that $x_1 \neq x_2$, $f_{M_1}(x_1) = f_{M_2}(x_1)$ and $f_{M_1}(x_2) = f_{M_2}(x_2)$; if M_1 and M_2 disagree for *all* inputs *etc*.

For the purposes of this question we regard f_{M_1} and f_{M_2} to be equal for some input x if either of the following is true:

- $f_{M_1}(x)$ and $f_{M_2}(x)$ are both undefined.
- $f_{M_1}(x)$ and $f_{M_2}(x)$ are both defined and have the same value.

For this question you must determine whether this decision problem is *solvable* or *unsolvable*. Your answer should have a clear statement of which of these two possibilities you think is the correct one, and a *proof* that justifies your decision. Simply stating 'this is solvable' or 'this is unsolvable' will gain little, if any, credit. The credit for answering this question is attached to your justification in the form of a proof.

The proof given can be in the less formal format used in the lectures[2].

8. $\star$ Repeat the previous question for the following decision problem.

   Instance Two Turing machines, M_1 and M_2, and an input alphabet Σ_I common to both machines.

   Question Let $L(M_1)$ and $L(M_2)$ be the languages semidecided by M_1 and M_2 respectively. Is it the case that $L(M_1) \cap L(M_2)$ is infinite and countable?

9. $\star$ Repeat question 7 again for the following decision problem.

   Instance: A TM M, an alphabet Σ_I of input symbols for M, and an arbitrary tape symbol σ.

   Question: Let $L \subseteq \Sigma_I^\star$ be the language semidecided by M. Let $S \subseteq L$ be the collection of strings in L containing an even number of occurrences of the symbol σ. Is S countably infinite?

10. $\star$ Repeat question 7 again for the following decision problem.

    The symmetric difference of two sets A and B is denoted $A + B$. It is defined as the collection of elements that are members of exactly one of the sets. That is, the elements that are members of either A or B, but not both. Consider the following decision problem:

[1] For the purposes of this question only we use a slightly modified definition for the output produced by a TM that has halted: it is the string that appears from position 1 on the tape up to the first blank on the tape.

[2] You may *not* use Rice's theorem (which is not covered in the course, but which can be found in most text books on the subject).

Instance: Two Turing machines, M_1 and M_2 with input alphabet $\{1\}$.

Question: Let $L(M_1)$ and $L(M_2)$ be the languages semidecided by M_1 and M_2 respectively. Does $L(M_1) + L(M_2)$ have size 1?

11. Explain how your answer to question 7 would be expressed in more detail in terms of formal languages.