Biabduction (and Related Problems) in Array Separation Logic

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Compositional proofs in separation logic (1)

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- Its compositional nature, the key to scalable analysis, is supported by two main pillars.
- The first pillar is the soundness of the following frame rule:

$$\frac{\{A\}\ C\ \{B\}}{\{A * F\}\ C\ \{B * F\}}\ \text{(Frame)}$$

where the separating conjunction $*$ is read, intuitively, as *"and separately in memory"*.

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$$\dfrac{\dfrac{\dfrac{\{A'\}\, C_1\, \{A\}}{\{A' * X\}\, C_1\, \{A * X\}}\ (\text{Frame})}{\{A' * X\}\, C_1\, \{B * Y\}}\ (\models) \qquad \dfrac{\{B\}\, C_2\, \{B'\}}{\{B * Y\}\, C_2\, \{B' * Y\}}\ (\text{Frame})}{\{A' * X\}\, C_1; C_2\, \{B' * Y\}}\ (;)$$

Symbolic-heap separation logic

- Terms t, pure formulas Π and spatial formulas F given by:

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t & ::= & x \in \mathsf{Var} \mid \mathsf{nil} \\
\Pi & ::= & t = t \mid t \neq t \mid \Pi \wedge \Pi \\
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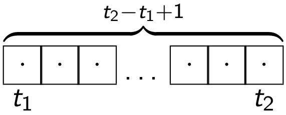

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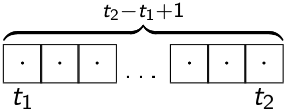

- We also allow linear arithmetic in the pure part.

Semantics of ASL

- Stacks are $s : \mathsf{Var} \to \mathsf{Val}$; heaps are $h : \mathsf{Loc} \rightharpoonup_{\mathrm{fin}} \mathsf{Val}$; $\circ$ is union of domain-disjoint heaps; e is the empty heap.

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s, h \models F_1 * F_2 & \Leftrightarrow & h = h_1 \circ h_2 \text{ and } s, h_1 \models F_1 \text{ and } s, h_2 \models F_2
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s, h \models \exists \mathbf{z}.\ \Pi : F & \Leftrightarrow & \exists \mathbf{v}.\ s[\mathbf{z} \mapsto \mathbf{v}], h \models \Pi \text{ and } s[\mathbf{z} \mapsto \mathbf{v}], h \models F
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Motivating example

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we get a valid spec $\{X\}\, C; \mathtt{foo(c,d)}\, \{Q * Y\}$.

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Spatially minimal, and incomparable, solutions include:

$$\begin{array}{l}
X := a = c \wedge b = d : \mathsf{emp} \quad \text{and} \quad Y := \mathsf{emp} \\
X := d < a : \mathsf{array}(c,d) \quad \text{and} \quad Y := \mathsf{array}(a,b) \\
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- Thus the problem is in NP.

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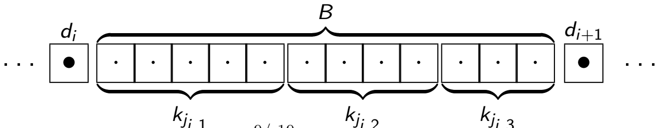

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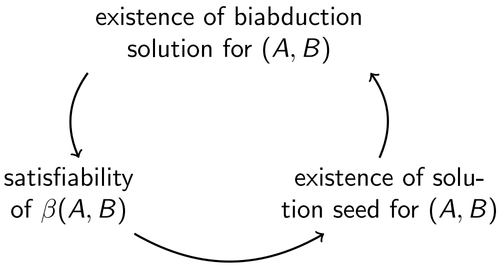

The formula $\beta(A, B)$

- Let (A, B) be an instance of the biabduction problem, where

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- This can be coded up as a Presburger formula $\beta(A, B)$, using the $\gamma(-)$ encoding of satisfiability.

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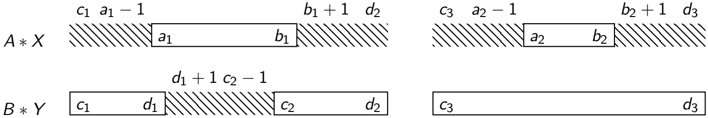

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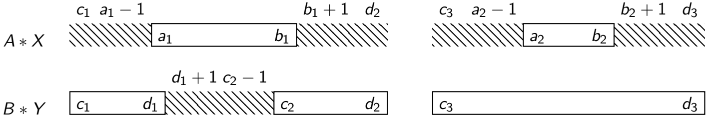

- We have to be a little careful about the pointer / array distinction though.

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- (Given G, we define A_G to encode a 3-colouring of the leaves, and B_G to encode a 3-colouring of G.)

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- Due to item 3, we can't allow $\exists$ over R-values in pointers.

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- I suspect the upper bound is closer to the "true complexity".

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- Extension of ASL with more expressive features (e.g. combine with list segments?).

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- We also establish decision procedures and complexity bounds for satisfiability and entailment.

Thanks for listening!

Paper available on arXiv:

`arXiv:1607.01993`