Complete Sequent Calculi for Induction and Infinite Descent

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Introduction

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  2. infinite descent *à la* Fermat.
- We work in first-order logic with inductive definitions.
- We formulate and compare proof-theoretic foundations of thes two styles of reasoning above, using Gentzen-style sequent calculus proof systems.

Part I

Inductive definitions in first-order logic

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- Our inductive rules are each of the form:

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- E.g., define N, E, O, R^+ (natural nos; even/odd nos; transitive closure of R) by rules

$$\begin{array}{rcl} & \Rightarrow & N0 \\ Nx & \Rightarrow & Nsx \end{array} \qquad \begin{array}{rcl} & \Rightarrow & E0 \\ Ox & \Rightarrow & Esx \\ Ex & \Rightarrow & Osx \end{array} \qquad \begin{array}{rcl} Rxy & \Rightarrow & R^+xy \\ R^+xy, R^+yz & \Rightarrow & R^+xz \end{array}$$

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- This least prefixed point can be approached via a sequence (φ^α_Φ) of approximants. E.g. for N we have:

$$\varphi^0_{\Phi_N} = \emptyset,\ \varphi^1_{\Phi_N} = \{0^M\},\ \varphi^2_{\Phi_N} = \{0^M, s^M 0^M\}, \ldots$$

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- A class of sets $\mathcal{H}$ over a first order structure M is a Henkin class if, roughly speaking, every first-order-definable relation is interpretable inside it.
- $(M, \mathcal{H})$ is a Henkin model if the least prefixed point of φ_Φ exists inside $\mathcal{H}$; we define P^M to be this point.

Part II

Sequent calculus for explicit induction

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NB. Mutual definitions give rise to mutual induction rules.

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- We then use this limit sequent to define a syntactic countermodel for $\Gamma \vdash \Delta$.
- (We need to define a Henkin class and deal with inductive predicates though.)

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Hence there is no elementary proof of cut-eliminability in LKID.

Part III

Sequent calculus for infinite descent

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- Pre-proofs are infinite (non-well-founded) derivation trees.
- For soundness we need to impose an additional condition on pre-proofs.

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- A trace progresses when the inductive predicate is unfolded using its case-split rule.
- A pre-proof is a proof if, for every infinite path in it, there is an infinitely progressing trace following some tail of the path.

A sample proof

$$
\dfrac{
\dfrac{
\dfrac{}{\vdash E0, O0}\,(ER_1)
}{x_0 = 0 \vdash Ex_0, Ox_0}\,(=\mathrm{L})
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\text{(etc.)}\\ \vdots\end{array}
}{{\color{red}Nx_1} \vdash Ex_1, Ox_1}\,(\text{Case } N)
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\dfrac{\dfrac{\dfrac{}{\vdash E0, O0}\,(ER_1)}{x_0 = 0 \vdash Ex_0, Ox_0}\,(=\text{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}(\text{etc.})\\ \vdots\end{array}}{Nx_1 \vdash Ex_1, Ox_1}\,(\text{Case } N)}{Nx_1 \vdash Ox_1, Osx_1}\,(OR_1)}{Nx_1 \vdash Esx_1, Osx_1}\,(ER_2)}{x_0 = sx_1, Nx_1 \vdash Ex_0, Ox_0}\,(=\text{L})}{Nx_0 \vdash Ex_0, Ox_0}\,(\text{Case } N)
$$

Continuing the expansion of the right branch, the formulas in red form an infinitely progressing trace, so the pre-proof thus obtained is indeed an LKID^{ω} proof.

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- Suppose $\Gamma \vdash \Delta$ is not valid. Since rules are locally sound, there must be an infinite path in the pre-proof consisting of invalid sequents.

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- By the soundness condition, there is an infinitely progressing trace of this path following some predicate P say.

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- By the soundness condition, there is an infinitely progressing trace of this path following some predicate P say.
- But then we can construct an infinite descending chain of ordinals based on the approximants of P, contradiction.

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- Given $\Gamma \vdash \Delta$ (not provable), we construct an infinite derivation tree corresponding to an exhaustive search for a proof of it.
- Either the tree gets stuck at some node which we call $\Gamma_\omega \vdash \Delta_\omega$, or else some branch fails the trace condition, in which case $\Gamma_\omega \vdash \Delta_\omega$ is the "limit union" of the sequents along this branch.

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- Either way, we show $\Gamma_\omega \vdash \Delta_\omega$ is not provable (this uses the trace condition).
- Thus we can use $\Gamma_\omega \vdash \Delta_\omega$ to construct a syntactic counter-model (the inductive predicate case also uses the trace condition).

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Unlike in LKID, cut-free proofs in LKID^{ω} enjoy a property akin to the subformula property, which seems close to the spirit of Girard's "purity of methods".

Part IV

Cyclic proofs by infinite descent

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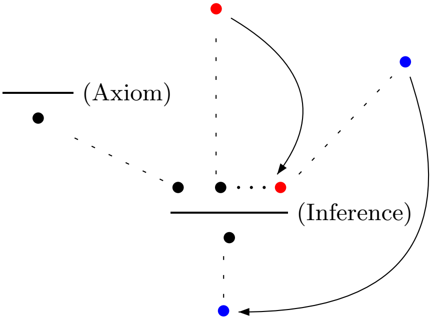
- However, the restriction of LKID$^\omega$ to proofs given by regular trees, which we call CLKID$^\omega$, is a natural one that *is* suitable for formal reasoning.
- In this restricted system, every proof can be represented as a finite (cyclic) graph.

Cyclic proofs



A cyclic proof

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$$

Any infinite path has a tail consisting of repetitions of the loop indicated by (†), and there is a progressing trace on this loop. By concatenating copies of this trace we obtain an infinitely progressing trace as required.

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Proposition (Proof-checking decidability)

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Any CLKID$^\omega$-provable sequent is also LKID-provable.

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This conjecture can be seen as a formalised version of:

> *Proof by induction is equivalent to regular proof by infinite descent.*

Part V

Summary

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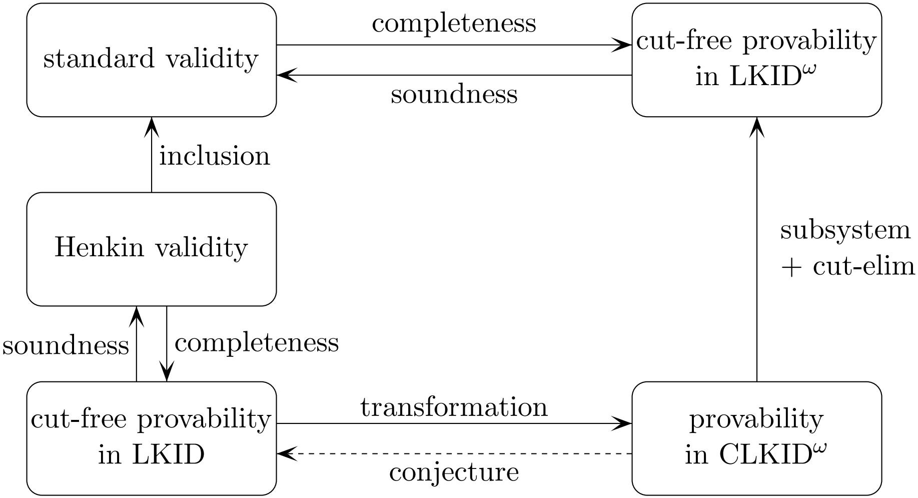

standard validity
completeness
soundness
cut-free provability in LKID^ω
inclusion
Henkin validity
subsystem + cut-elim
soundness
completeness
cut-free provability in LKID
transformation
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- Cyclic systems have been developed for various other logics with inductive definitions or fixed point operators.
- Attempts at solving the conjecture...

Further reading

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