# *An Introduction to Cyclic Proofs*

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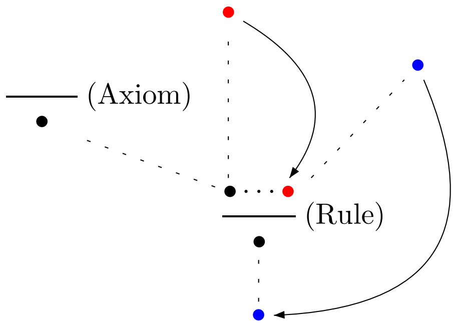
PPLV seminar, 28th Oct 2022

## *Cyclic pre-proofs*

A cyclic pre-proof is a derivation tree with a backlink from each open leaf (“bud”) to an identical “companion”:



Cyclic proof = pre-proof $\mathcal{P}$ + soundness condition $S(\mathcal{P})$.

## *An invalid pre-proof*

$$\begin{tikzcd}
\vdash \bot \arrow[d, dash, "\text{(Weak)}"] \\
\vdash \bot, \bot \arrow[d, dash, "\text{(Contr)}"] \\
\vdash \bot \arrow[uu, dashed, bend left=90, leftarrow]
\end{tikzcd}$$

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- This is certainly a pre-proof, but obviously it cannot be accepted as valid!

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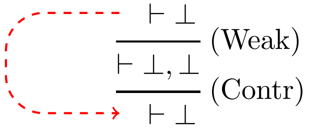

- This is certainly a pre-proof, but obviously it cannot be accepted as valid!
- Here, we formed a cycle but failed to make any appreciable “progress”.

## *The need for a soundness condition*

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- When proofs are finite trees, this guarantees that any provable judgement is valid: supposing not, then some axiom in the tree must be invalid, contradiction.
- However, when proofs are cyclic graphs, local soundness just says that if the root judgement is invalid then there is an infinite path of invalid judgements in the tree.

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- In any reasonable proof system the rules must be locally sound: if all premises of the rule are valid then so is its conclusion.
- When proofs are finite trees, this guarantees that any provable judgement is valid: supposing not, then some axiom in the tree must be invalid, contradiction.
- However, when proofs are cyclic graphs, local soundness just says that if the root judgement is invalid then there is an infinite path of invalid judgements in the tree.
- A soundness condition for cyclic proofs must therefore rule out the existence of such paths.

## *Infinite descent*

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*If there were any integral right triangle that had an area equal to a square, there would be another triangle less than that one which would have the same property...*

*Now it is the case that, given a number, there are not infinitely many numbers less than that one in descending order ... Whence one concludes that it is therefore impossible that there be any right triangle of which the area is a square...*

Pierre de Fermat, *Relation des nouvelles d'ecouvertes en la science des nombres*, letter to Pierre de Carcavi, 1659

## *Infinite descent example*

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$\sqrt{2}$ *is not rational.*

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Define $x' = 2y - x$ and $y' = x - y$. Then $x'/y' = \sqrt{2}$. Now observe that $1 < x^2/y^2 < 4$, so $y < x < 2y$, and so $0 < y' < y$.

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1. By supposition there are no infinite *tick* sequences from $Cl$. However, the infinite path *does* create such an infinite sequence, since $(\langle tick \rangle)$ is applied infinitely often.

2. There must be some ordinal-indexed overapproximation of the fixed point $\nu^\alpha X.\, \langle tick \rangle X$ of which $Cl$ is not a member. Unfolding $\nu X$ infinitely often (by $(\nu)$) creates an infinite descending chain of such ordinals, from $\alpha$ — but these are well-founded.

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Then $\{P\}\,C\,\{Q\}$ is valid when:

$$\text{if } \sigma \models P \text{ and } \langle C, \sigma\rangle \to^* \langle \sigma'\rangle \text{ then } \sigma' \models Q\ .$$

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while i>0 {if * then i--;};
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But program commands are symbolically executed infinitely often along this path. Thus the assumed execution from $\langle C, \sigma\rangle$ is in fact infinite: contradiction.

## *Inductive definitions in first-order logic*

Consider these inductive definitions of predicates $N, E, O$:

$$\begin{array}{rcl} & \Rightarrow & N0 \\ Nx & \Rightarrow & Nsx \end{array} \qquad \begin{array}{rcl} & \Rightarrow & E0 \\ Ex & \Rightarrow & Osx \\ Ox & \Rightarrow & Esx \end{array}$$

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These definitions generate case-split rules, e.g., for $N$:

$$\frac{\Gamma, t = 0 \vdash \Delta \qquad \Gamma, t = sx, Nx \vdash \Delta}{\Gamma, Nt \vdash \Delta}\ (\text{Case } N)$$

(where $x$ is fresh).
Note that $Nx$ in the right-hand premise is obtained by *unfolding* $Nt$ in the conclusion.

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}{
\dfrac{Nx \vdash Ex, Ox}{Nx \vdash Ex \vee Ox}\,(\vee)
}\,(\text{Case } N)
$$

Note that here we examine formulas on the left of the turnstile!

## *Explanation of soundness*

Suppose that $Nx \vdash Ex \vee Ox$ is invalid, meaning that $M \models_\rho Nx$ (for some structure $M$ and valuation $\rho$) but $M \not\models_\rho Ex \vee Ox$.

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1. that $[\![N]\!]_M$ is a well-founded set and we have an infinite descent in these "numerals", from $\rho(x)$, because of the infinite unfolding of $Nx$; or
2. that if $\rho(x) \in [\![N]\!]_M$ that it is a member of some underapproximation $[\![N]\!]^\alpha_M$, and we have an infinite descent in these approximant ordinals, again because of the infinite unfolding of $N$.

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$$
\begin{prooftree}
\AxiomC{}
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\RightLabel{$(=)$}
\UnaryInfC{$x = 0 \vdash Nx$}
\AxiomC{$Ox \vdash Nx$}
\RightLabel{(Subst)}
\UnaryInfC{$Oy \vdash Ny$}
\RightLabel{$(N)$}
\UnaryInfC{$Oy \vdash Nsy$}
\RightLabel{$(=)$}
\UnaryInfC{$x = sy, Oy \vdash Nx$}
\RightLabel{(Case $E$)}
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\RightLabel{$(N)$}
\UnaryInfC{$Ey \vdash Nsy$}
\RightLabel{$(=)$}
\UnaryInfC{$x = sy, Ey \vdash Nx$}
\RightLabel{(Case $O$)}
\UnaryInfC{$Ox \vdash Nx$}
\RightLabel{$(\vee)$}
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\end{prooftree}
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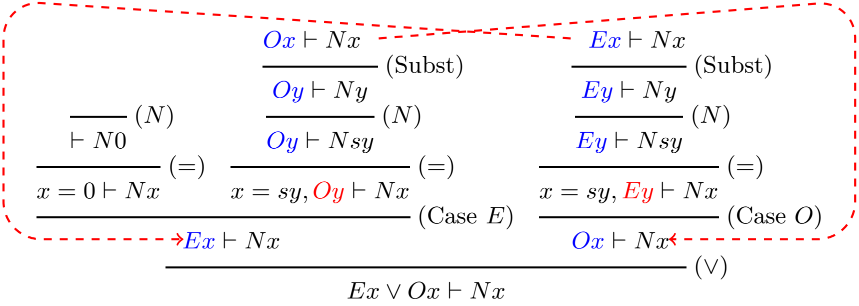
Here's a proof of the converse statement, $Ex \vee Ox \vdash Nx$.



Soundness justification is as before.

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Dealing with this is essentially a matter of book-keeping. And it might not be necessary if there are no tricky rules.

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- A trace along a path in a pre-proof is obtained by chaining trace pairs together in the obvious way.
- A trace is infinitely progressing if it contains infinitely many progressing trace pairs.

## *A general soundness condition*

Given some appropriate[1] notion of "trace pairs" for a cyclic proof system, one can then state a general soundness condition:

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Virtually all the cyclic systems I know use a condition of this form, or which can be rewritten as such.

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## *Two relevant facts*

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2. It is decidable whether a pre-proof $\mathcal{P}$ is a cyclic proof or not. Build two Büchi automata: $B_1$ accepting all infinite paths in $\mathcal{P}$; and $B_2$ accepting all paths with an infinitely progressing trace on some tail. The soundness condition amounts to checking $\mathcal{L}(B_1) \subseteq \mathcal{L}(B_2)$.

## *Some logics with cyclic proof systems*

- $\mu$-calculus (modal, first-order, process verification)
- temporal logic (CTL, LTL,...)
- first-order logic with ind. defns
- separation logic with ind. defns
- Hoare logic and variants (e.g. termination)
- linear logic with fixed points
- modal logic (of certain kinds)
- Kleene algebra
- combinations of the above

This is by no means a complete list!

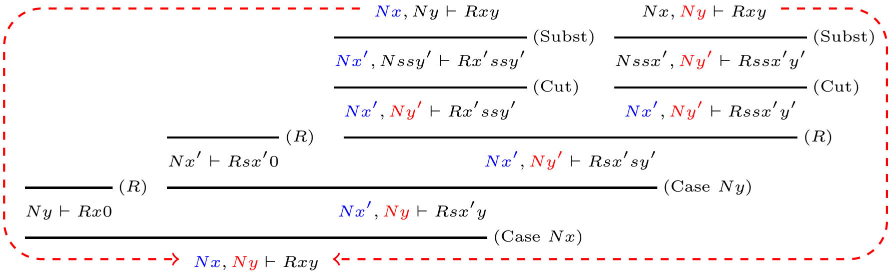
*Thanks!*

## *Failure of per-cycle soundness*

Consider inductive definitions:

$$\begin{array}{rcl} & \Rightarrow & N0 \\ Nx & \Rightarrow & Nsx \end{array} \qquad \begin{array}{rcl} & \Rightarrow R0y & \Rightarrow Rx0 \\ R(ssx,y), R(x,ssy) & \Rightarrow & Rsxsy \end{array}$$

Now $Nx, Ny \vdash Rxy$ is not valid. E.g. $R(s0, ss0)$ fails. But:



## *The most common question*

Infinite descent principle for $\mathbb{N}$:

$$\frac{\neg P(k) \to (\exists k' < k \in \mathbb{N}.\ \neg P(k'))}{\forall n \in \mathbb{N}.\ P(n)}\ (k \text{ arbitrary})$$

Complete induction principle:

$$\frac{(\forall k' < k \in \mathbb{N}.\ P(k')) \to P(k)}{\forall n \in \mathbb{N}.\ P(n)}\ (k \text{ arbitrary})$$

These are obviously interderivable, so aren't cyclic proof and induction proof just the same thing?

The main difficulty is that

- cyclic proof encodes a relatively strong form of infinite descent that is implicit in the structure of the proof (nested cycles, etc.), while

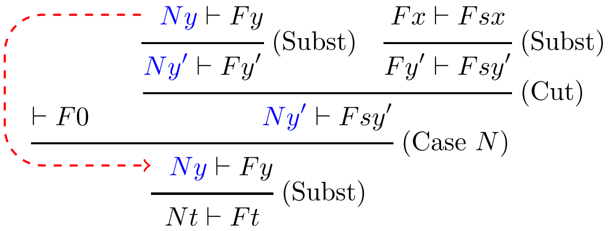
- induction proof often uses a relatively weak form of induction encoded explicitly as a local inference rule. E.g., for $N$:

$$\frac{\vdash F0 \qquad Fx \vdash Fsx}{Nt \vdash Ft}\,(\text{Ind } N)$$

The equivalence of the two styles, for $FOL$ with ind defns, was a conjecture (Brotherston and Simpson, LICS 2007)

## *From cyclic to induction proof*

Cyclic derivation of $N$-induction:



This construction generalises to arbitrary inductive definitions.

*Theorem*

*Any sequent provable by induction also has a cyclic proof.*

## *Peano arithmetic using inductive defns*

There is an embedding of Peano arithmetic ($PA$) into an explicit-induction proof system:

- add the first six Peano axioms as closed formulas (on the LHS);
- add formulas $Nx$ for each free variable $x$;
- relativise all quantifiers over $N$;
- the Peano induction axiom follows from the induction rule for $N$.

This means we can formalise $PA$ in a cyclic proof system as well.

## *An aside on completeness*

If we allow proofs to be arbitrary infinite trees rather than cyclic graphs then the system becomes complete (Brotherston and Simpson LICS 2007).

Since we can formalise $PA$ using induction and thus cyclic proof, this gives us a complete system for arithmetic.

However, since true arithmetic is not even semidecidable, there can be no recursive enumeration of the proofs in this system!

## *Results on cyclic arithmetic*

*Theorem (Simpson, FoSSaCS 2017)*

*Cyclic arithmetic is equivalent to Peano arithmetic.*

Proof is by formalising the soundness of cyclic arithmetic inside $ACA_0$ which is conservative over $PA$.

*Theorem (Berardi and Tatsuta, LICS 2017)*

*Cyclic proof is equivalent to induction proof for any signature that includes Peano arithmetic.*

Proof is by explicit conversion, defining a notion of ¡ for all predicates and formalising a version of Ramsay's theorem using explicit induction.

## *However. . .*

*Theorem (Berardi and Tasuta, FoSSaCS 2017)*

*There is a signature for which cyclic proof is not equivalent to induction proof.*

This is essentially because cyclic proof implicitly lets us do things like infinite descent over the max or min of two numbers, concepts which might not be explicitly formalisable in restricted signatures.

## CYCLIST *theorem prover*

- A generic (logic-independent) theorem prover that supports cyclic proof
- Lead developer Nikos Gorogiannis (Facebook & U. Middlesex)
- Support for inductive definitions
- Automatic checking of cyclic soundness condition (using the Büchi automata construction from yesterday)
- Open source:

`github.com/ngorogiannis/cyclist`

## *Some* CYCLIST *instantiations*

- first-order logic with ind defns
- separation logic with ind defns
- Hoare logic for program termination with recursive procedures (R. Rowe)
- Hoare logic for temporal program properties (G. Tellez Espinosa)

## *Build your own* CYCLIST *instantiation*

To implement your favourite cyclic proof system in CYCLIST you need to provide the following (to `Ocaml` functors):

- a syntax for proof judgements;
- some proof rules for judgements;
- the (progressing) trace pairs associated with each proof rule;
- a matching condition for backlinking;
- (optional) a preferred search strategy.

Why not try it?