Bunched Logics Displayed

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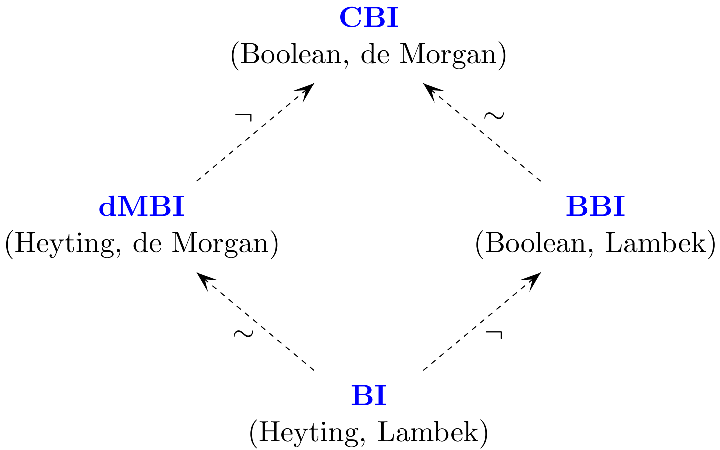
Part I

Motivation

Bunched logic

- A variety of substructural logic (but more like relevant logic than linear logic);
- Obtained by combining an additive propositional logic with a multiplicative one;
- Naive reading of additives leads to a natural resource interpretation of formulas (used e.g. in separation logic);
- Original bunched logic is O'Hearn and Pym's BI, which is essentially IL + MILL;
- One can think of additive and multiplicative components as each being either classical or intuitionistic.

The bunched logic family



- Subtitles (X,Y) indicate the underlying algebras.
- Arrows denote addition of classical negations $\neg$ or $\sim$.

LBI: *the* BI *sequent calculus*

- Formulas of BI are given by additive connectives $\top, \bot, \vee, \wedge, \rightarrow$ of IL plus multiplicative $\top^*$, $*$ and $\mathrel{-\!\!*}$;
- Sequents are $\Gamma \vdash F$ where F a formula and Γ a bunch:

$$\Gamma ::= F \mid \emptyset \mid \varnothing \mid \Gamma\,;\Gamma \mid \Gamma\,,\Gamma$$

- Rules for $\mathrel{-\!\!*}$ are:

$$\frac{\Delta \vdash F_1 \quad \Gamma(F_2) \vdash F}{\Gamma(\Delta\,, F_1 \mathrel{-\!\!*} F_2) \vdash F}\,(\mathrel{-\!\!*}\mathrm{L}) \qquad \frac{\Gamma\,, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}\mathrm{R})$$

  where $\Gamma(\Delta)$ is bunch Γ with sub-bunch Δ;
- LBI satisfies cut-elimination (Pym 02).

Extending the BI *sequent calculus*

- Consider BBI obtained by adding additive classical negation $\neg$ to BI;
- We need multiple conclusions in some form to have cut-free proofs of e.g. $\neg\neg F \vdash F$;
- But the multiplicative rules do not behave well with multiple conclusions, e.g.:

$$\frac{\Gamma \vdash F \quad \Gamma' \vdash G}{\Gamma\,,\,\Gamma' \vdash F * G}\,(*\mathrm{R}) \qquad\qquad \frac{\Gamma \vdash F\,;\,\Delta \quad \Gamma' \vdash G\,;\,\Delta}{\Gamma\,,\,\Gamma' \vdash F * G; \Delta}\,(?)$$

Sound *Unsound*!

- Similar remarks apply to dMBI and CBI.
- We take a different approach, based on display calculi.

Part II

From elementary logics to bunched logics

Elementary components of bunched logic

Additives:	$\top$	$\bot$	$\neg$	$\vee$	$\wedge$	$\rightarrow$
Multiplicatives:	$\top^*$	$\bot^*$	$\sim$	$\overset{*}{\vee}$	$*$	$-\!\!*$

- IL and CL are standard intuitionistic / classical logic over the additives;
- LM and dMM are Lambek / de Morgan logic over the multiplicatives;
- Each of the above logics $\mathcal{L}$ given by a Hilbert-style proof system $\mathrm{HL}_{\mathcal{L}}$.
- Define "elementary logics" $\mathcal{E} = \{\mathrm{IL}, \mathrm{CL}, \mathrm{LM}, \mathrm{dMM}\}$.

Hilbert presentations of LM *and* dMM

$$F \vdash F \qquad\qquad F * \top^* \dashv\vdash F$$

$$F * (G * H) \dashv\vdash (F * G) * H \qquad F * G \vdash G * F$$

$$\frac{F_1 \vdash G_1 \quad F_2 \vdash G_2}{F_1 * F_2 \vdash G_1 * G_2} \quad \frac{F * G \vdash H}{F \vdash G \mathrel{-\!\!*} H} \quad \frac{F \vdash G \mathrel{-\!\!*} H}{F * G \vdash H} \quad \frac{F \vdash G \quad G \vdash H}{F \vdash H}$$

....

$$\bot^* \dashv\vdash {\sim}\top^* \qquad F \stackrel{*}{\vee} G \dashv\vdash {\sim}({\sim}F * {\sim}G)$$

$${\sim}F \dashv\vdash F \mathrel{-\!\!*} \bot^* \qquad\qquad {\sim}{\sim}F \vdash F$$

- The axioms below the line are present in $\mathrm{HL}_{\mathrm{dMM}}$ only.

Definition of bunched logics

- We define the bunched logics $\mathcal{B} = \{\mathrm{BI}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$ in terms of $\mathcal{E}$ as follows:

$$\begin{array}{rcl} \mathrm{BI} & = & \mathrm{IL} + \mathrm{LM} \\ \mathrm{BBI} & = & \mathrm{CL} + \mathrm{LM} \\ \mathrm{dMBI} & = & \mathrm{IL} + \mathrm{dMM} \\ \mathrm{CBI} & = & \mathrm{CL} + \mathrm{dMM} \end{array}$$

  where + is interpreted as union of Hilbert presentations.
- These definitions are taken as the baseline for correctness - no semantics in this approach!
- However, our definitions agree with others in the literature.

Part III

Display calculus fundamentals

Syntax of display calculi

- Structures are constructed from formulas using structural connectives:

Additive	*Multiplicative*	*Arity*	*Antecedent*	*Consequent*
$\emptyset$	$\varnothing$	0	truth	falsity
$\sharp$	$\flat$	1	negation	negation
$;$	$,$	2	conjunction	disjunction
$\Rightarrow$	$\multimap$	2	–	implication

- Consecutions are given by $X \vdash Y$ for X, Y structures.
- We classify substructures of $X \vdash Y$ as antecedent or consequent parts (similar to positive / negative occurrences in formulas).

Display-equivalence

Display calculi are characterised by the availability of a display-equivalence on consecutions:

Definition

The least congruence $\equiv_D$ generated by a set of display postulates of the form $X \vdash Y <>_D X' \vdash Y'$ is a display-equivalence iff:

- for any antecedent part Z of $X \vdash Y$ there is a W s.t. $X \vdash Y \equiv_D Z \vdash W$;
- for any consequent part Z of $X \vdash Y$ there is a W s.t. $X \vdash Y \equiv_D W Z$.

Specifying display calculi

A display calculus $\mathrm{DL}_{\mathcal{L}}$ for a logic $\mathcal{L}$ is specified by:

- A set each of antecedent and consequent structural connectives;
- Display postulates generating a display-equivalence;
- Logical rules for the logical connectives;
- Structural rules for the structural connectives.
- Standard rules:

$$\frac{}{P \vdash P}\,(\mathrm{Id}) \qquad \frac{X \vdash F \quad F \vdash Y}{X \vdash Y}\,(\mathrm{Cut}) \qquad \frac{X' \vdash Y'}{X \vdash Y}\; X \vdash Y \equiv_D X' \vdash Y' \;(\mathrm{D}\equiv)$$

Belnap '82 gives a set of syntactic conditions over proof rules which guarantee cut-elimination for any display calculus.

Part IV

Display calculi for bunched logics

$\mathrm{DL}_{\mathrm{IL}}$: *a display calculus for* IL

Antecedent structure connectives: $\emptyset \quad ;$

Consequent structure connectives: $\Rightarrow$

Display postulates: $X \,;\, Y \vdash Z \;\; <>_D \;\; X \vdash Y \Rightarrow Z \;\; <>_D \;\; Y \,;\, X \vdash Z$

Logical rules:

$$\frac{}{\bot \vdash X}\,(\bot L) \qquad \frac{\emptyset \vdash X}{\top \vdash X}\,(\top L) \qquad \frac{}{X \vdash \top}\,(\top R)$$

$$\frac{F \,;\, G \vdash X}{F \wedge G \vdash X}\,(\wedge L) \qquad \frac{F \vdash X \quad G \vdash X}{F \vee G \vdash X}\,(\vee L) \qquad \frac{X \vdash F \quad G \vdash Y}{X \,;\, F \to G \vdash Y}\,(\to L)$$

$$\frac{X \vdash F \quad X \vdash G}{X \vdash F \wedge G}\,(\wedge R) \qquad \frac{X \vdash F_i}{X \vdash F_1 \vee F_2}\; i \in \{1,2\}\;(\vee R) \qquad \frac{X \,;\, F \vdash G}{X \vdash F \to G}\,(\to R)$$

Structural rules:

$$\frac{\emptyset \,;\, X \vdash Y}{X \vdash Y}\,(\emptyset L) \qquad \frac{W \,;\, (X \,;\, Y) \vdash Z}{(W \,;\, X) \,;\, Y \vdash Z}\,(AAL) \qquad \frac{X \vdash Z}{X \,;\, Y \vdash Z}\,(WkL) \qquad \frac{X \,;\, X \vdash Y}{X \vdash Y}\,(CtrlL)$$

$\mathrm{DL}_{\mathrm{dMM}}$: *a display calculus for* dMM

Antecedent structure connectives: $\varnothing \quad \flat \quad ,$

Consequent structure connectives: $\varnothing \quad \flat \quad ,$

Display postulates:

$$\begin{array}{ccccc} X , Y \vdash Z & <>_D & X \vdash \flat Y , Z & <>_D & Y , X \vdash Z \\ X \vdash Y , Z & <>_D & X , \flat Y \vdash Z & <>_D & X \vdash Y , Z \\ X \vdash Y & <>_D & \flat Y \vdash \flat X & <>_D & \flat\flat X \vdash Y \end{array}$$

Logical rules:

$$\begin{array}{c} \varnothing \vdash X \\ \hline \top^* \vdash X \end{array} (\top^* \mathrm{L}) \qquad \begin{array}{c} \\ \hline \varnothing \vdash \top^* \end{array} (\top^* \mathrm{R}) \qquad \begin{array}{c} \\ \hline \bot^* \vdash \varnothing \end{array} (\bot^* \mathrm{L}) \qquad \begin{array}{c} X \vdash \varnothing \\ \hline X \vdash \bot^* \end{array} (\bot^* \mathrm{R})$$

$$\begin{array}{c} F , G \vdash X \\ \hline F * G \vdash X \end{array} (*\mathrm{L}) \qquad \begin{array}{c} X \vdash F \quad Y \vdash G \\ \hline X , Y \vdash F * G \end{array} (*\mathrm{R}) \qquad \begin{array}{c} F \vdash X \quad G \vdash Y \\ \hline F \mathbin{\overset{*}{\vee}} G \vdash X , Y \end{array} (\overset{*}{\vee}\mathrm{L}) \qquad \begin{array}{c} X \vdash F , G \\ \hline X \vdash F \mathbin{\overset{*}{\vee}} G \end{array} (\overset{*}{\vee}\mathrm{R})$$

$$\begin{array}{c} \flat F \vdash X \\ \hline \sim F \vdash X \end{array} (\sim \mathrm{L}) \qquad \begin{array}{c} X \vdash \flat F \\ \hline X \vdash \sim F \end{array} (\sim \mathrm{R}) \qquad \begin{array}{c} X \vdash F \quad G \vdash Y \\ \hline X , F \mathrel{-\!\!*} G \vdash Y \end{array} (-\!\!*\mathrm{L}) \qquad \begin{array}{c} X , F \vdash G \\ \hline X \vdash F \mathrel{-\!\!*} G \end{array} (-\!\!*\mathrm{R})$$

Structural rules:

$$\begin{array}{c} \varnothing , X \vdash Y \\ \hline\hline X \vdash Y \end{array} (\varnothing\mathrm{L}) \qquad \begin{array}{c} W , (X , Y) \vdash Z \\ \hline\hline (W , X) , Y \vdash Z \end{array} (\mathrm{MAL}) \qquad \begin{array}{c} X \vdash Y , \varnothing \\ \hline\hline X \vdash Y \end{array} (\varnothing\mathrm{R}) \qquad \begin{array}{c} W \vdash (X , Y) , Z \\ \hline\hline W \vdash X , (Y , Z) \end{array} (\mathrm{MAR})$$

Display calculi for bunched logics

We obtain display calculi $\mathrm{DL}_{\mathcal{L}}$ for each $\mathcal{L} \in \mathcal{B}$ by setting for $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{E}$:

$$\mathrm{DL}_{\mathcal{L}_1+\mathcal{L}_2} = \mathrm{DL}_{\mathcal{L}_1} + \mathrm{DL}_{\mathcal{L}_2}$$

where $+$ is component-wise union of display calculus specifications.

Lemma

For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$ the least congruence induced by the display postulates of $\mathrm{DL}_{\mathcal{L}}$ is a display-equivalence for $\mathrm{DL}_{\mathcal{L}}$.

Proof.

Easy verification for each $\mathcal{L} \in \mathcal{E}$. The result then follows for each $\mathcal{L} \in \mathcal{B}$. □

Principal results

Theorem (Cut-elimination)

For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, any $\mathrm{DL}_{\mathcal{L}}$ proof of $X \vdash Y$ can be transformed into a (Cut)-free proof of $X \vdash Y$.

Proof.

Verify that Belnap's conditions C1–C8 hold of $\mathrm{DL}_{\mathcal{L}}$ for each $\mathcal{L} \in \mathcal{E}$, whence it follows that the same conditions must hold also for each $\mathcal{L} \in \mathcal{B}$. □

Theorem (Soundness / Completeness)

For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, $X \vdash Y$ is $\mathrm{DL}_{\mathcal{L}}$-provable just in case it is valid wrt. the Hilbert presentation of $\mathcal{L}$.

Proof.

Straightforward verification for each $\mathcal{L} \in \mathcal{E}$, whence the result follows directly for each $\mathcal{L} \in \mathcal{B}$. □

Part V

Consequences

Translating LBI *into* $\mathrm{DL}_{\mathrm{BI}}$

Recall the LBI rules for $-\!*$:

$$\frac{\Delta \vdash F_1 \quad \Gamma(F_2) \vdash F}{\Gamma(\Delta \;,\; F_1 -\!* F_2) \vdash F}\,(-\!*\mathrm{L}) \qquad\qquad \frac{\Gamma \;,\; F \vdash G}{\Gamma \vdash F -\!* G}\,(-\!*\mathrm{R})$$

$(-\!*\mathrm{R})$ has a direct equivalent in $\mathrm{DL}_{\mathrm{BI}}$, while $(-\!*\mathrm{L})$ can be derived in $\mathrm{DL}_{\mathrm{BI}}$ as follows:

$$\dfrac{\dfrac{\Delta \vdash F_1 \qquad \dfrac{\Gamma(F_2) \vdash F}{F_2 \vdash X}\,(\mathrm{D}\equiv)}{\Delta \;,\; F_1 -\!* F_2 \vdash X}\,(-\!*\mathrm{L})}{\Gamma(\Delta \;,\; F_1 -\!* F_2) \vdash F}\,(\mathrm{D}\equiv)$$

Translation preserves cut-freeness of proofs.

Translating $\mathrm{DL}_{\mathrm{BI}}$ *into* LBI

For any consecution $X \vdash Y$ define its display-normal form $\ulcorner X \vdash Y \urcorner$ as the result of iteratively applying transformations:

$$\begin{array}{rcl} X \vdash Y \Rightarrow Z & \mapsto & X \, ; Y \vdash Z \\ X \vdash Y \multimap Z & \mapsto & X \, , Y \vdash Z \end{array}$$

Then $\ulcorner X \vdash Y \urcorner$ is always an LBI sequent and the rules of $\mathrm{DL}_{\mathrm{BI}}$ are LBI-derivable under translation $\ulcorner - \urcorner$, e.g.:

$$\frac{\ulcorner X \vdash F \urcorner \quad \ulcorner G \vdash Y \urcorner}{\ulcorner X \, , F \multimap\!\!* \, G \vdash Y \urcorner} \quad = \quad \frac{X \vdash F \quad \Gamma(G) \vdash H}{\Gamma(X \, , F \multimap\!\!* \, G) \vdash H}$$

Translation again preserves cut-freeness of proofs, so we have:

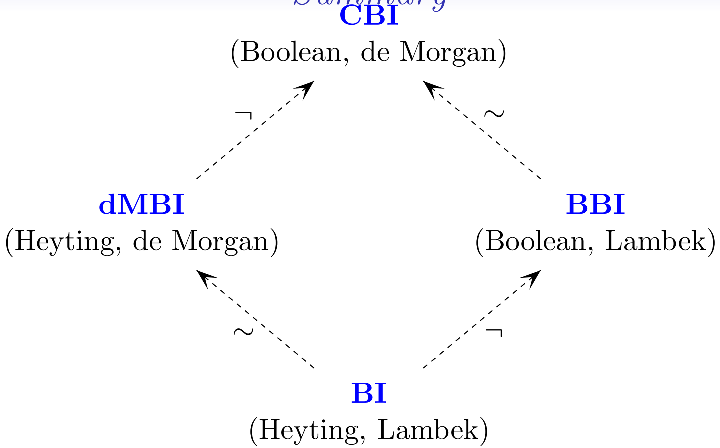
Proposition

Cut-elimination in LBI $\Leftrightarrow$ *cut-elimination in* $\mathrm{DL}_{\mathrm{BI}}$.

Failure of general sequent translation

- So, LBI can be seen as an optimised version of $\mathrm{DL_{BI}}$;
- However, the display calculi for the other bunched logics do not optimise in the same way due to their use of structural negations $\sharp$ and $\flat$.
- E.g., in the case of BBI, how do we define the display-normal form $\ulcorner F \,,\, \sharp G \vdash H \urcorner$?
- We suggest that there is really no sensible way of defining a “bunched sequent” representation of such consecutions.
- Thus we argue that our display calculi are really canonical proof systems for the bunched logics.

Summary



- Relevant logic connection: dMBI can now be seen to be a conservative extension of RW.
- Restall's decidability techniques for display calculi seem certain to apply to both dMBI and BI.

Summary

- All four bunched logics now have a sound, complete and cut-eliminating proof theory.
- Our development aims to maximally exploit the symmetries and common elements of the setting.
- We argue that our display calculi are really canonical proof systems for the bunched logics.
- In particular, our display calculus for BI is really a simple reformulation of the BI sequent calculus.

Further reading

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