Craig interpolation in displayable logics

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Craig interpolation

Definition

A (propositional) logic satisfies Craig interpolation iff for any provable $F \vdash G$ there exists an interpolant I s.t.:

$$F \vdash I \text{ provable and } I \vdash G \text{ provable and } \mathcal{V}(I) \subseteq \mathcal{V}(F) \cap \mathcal{V}(G)$$

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- computer science: invariant generation; type inference; model checking; ontology decomposition

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- But decidability, interpolation etc. don't follow directly as they often do in sequent calculi.
- We show interpolation for a large class of display calculi.

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- Consecutions are given by $X \vdash Y$ for X, Y structures.
- Substructures of $X \vdash Y$ are antecedent or consequent parts (similar to positive / negative occurrences in formulas).

Display-equivalence

We have the following display postulates:

$$\begin{array}{ccccc} X ; Y \vdash Z & <>_D & X \vdash \sharp Y ; Z & <>_D & Y ; X \vdash Z \\ X \vdash Y ; Z & <>_D & X ; \sharp Y \vdash Z & <>_D & X \vdash Z ; Y \\ X \vdash Y & <>_D & \sharp Y \vdash \sharp X & <>_D & \sharp\sharp X \vdash Y \end{array}$$

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Proposition (Display property)

For any antecedent part Z of $X \vdash Y$ there is a W s.t.

$$X \vdash Y \equiv_D Z \vdash W$$

(and similarly for consequent parts).

Some proof rules

Identity rules:

$$\frac{}{P \vdash P}\,(\mathrm{Id}) \qquad \frac{X' \vdash Y'}{X \vdash Y}\ (X \vdash Y \equiv_D X' \vdash Y')\ (\equiv_D)$$

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Logical rules:

$$\frac{F \,;\, G \vdash X}{F\&G \vdash X}\,(\&\mathrm{L}) \qquad \frac{X \vdash F \quad Y \vdash G}{X \,;\, Y \vdash F\&G}\,(\&\mathrm{R}) \qquad \ldots$$

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Structural rules:

$$\frac{W\,;(X\,;Y) \vdash Z}{(W\,;X)\,;Y \vdash Z}\,(\alpha) \qquad \frac{\emptyset\,;X \vdash Y}{X \vdash Y}\,(\emptyset \mathrm{C_L})$$

$$\frac{X \vdash Z}{X\,;Y \vdash Z}\,(\mathrm{W}) \qquad \frac{X\,;X \vdash Y}{X \vdash Y}\,(\mathrm{C}) \qquad \ldots$$

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- Induction on proofs: from interpolants for the premises of a rule, construct an interpolant for its conclusion.
- But not enough info to do this for display steps, e.g.:

$$\frac{X \,;\, Y \vdash Z}{X \vdash \sharp Y \,;\, Z}\,(\equiv_D)$$

Local AD-interpolation (LADI) property

Let $\equiv_{AD}$ be the least equivalence closed under $\equiv_D$ and applications of associativity (α) (if present).

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Definition

A proof rule with conclusion $\mathcal{C}$ has the LADI property if, given that for each premise of the rule $\mathcal{C}_i$ we have interpolants for all $\mathcal{C}'_i \equiv_{AD} \mathcal{C}_i$, we can construct interpolants for all $\mathcal{C}' \equiv_{AD} \mathcal{C}$.

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Proposition

If the proof rules of a display calculus $\mathcal{D}$ all have the LADI property then $\mathcal{D}$ enjoys Craig interpolation.

LADI: (&R)

$$\frac{X \vdash F \quad Y \vdash G}{X \,;\, Y \vdash F \& G}\ (\&\mathrm{R})$$

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Main issue: show $I \vdash Z$ provable given $I \vdash U$ provable.

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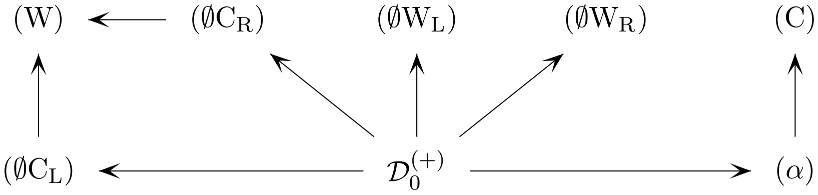
If not, about the best we can do is:

$$X_1 \vdash \sharp X_2; (\sharp(X_1; X_2); Y)$$

whose interpolant is far too weak to work for $X_1 \vdash \sharp X_2; Y$.

Summary of results



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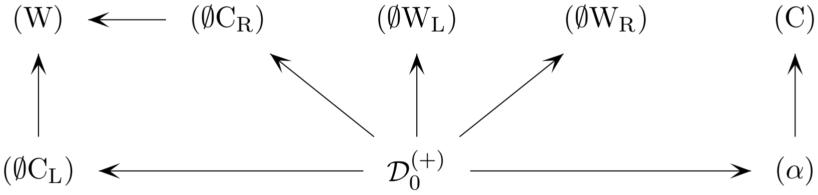

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Theorem

Any display calculus satisfying the constraints in the above diagram has Craig interpolation.
(This includes MLL, MALL and classical logic.)

Future work

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   - modalities, quantifiers, linear exponentials ...

Further reading

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