Classical BI

(A logic for reasoning about dualising resources)

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Boolean BI
(O'Hearn and Pym '99)

- A substructural logic with natural resource interpretation.
- Formula connectives:

$$\begin{array}{lllllll} \textit{Additive:} & \top & \bot & \neg & \wedge & \vee & \rightarrow \\ \textit{Multiplicative:} & \top^* & & & * & & -\!\!* \end{array}$$

- Additives are interpreted classically.

Resource models of BBI

- Models of BBI are relational commutative monoids $\langle R, \circ, e \rangle$ (we assume $\circ$ a partial function), where:

  R: a set of resources
  $\circ$: a way of (partially) combining resources
  e: the distinguished empty resource

- Separation logic is based on a BBI-model of heaps.
- Multiplicative formulas talk about resources $r \in R$:

$$
\begin{array}{rcl}
r \models \top^* & \Leftrightarrow & r = e \\
r \models F_1 * F_2 & \Leftrightarrow & r = r_1 \circ r_2 \text{ and } r_1 \models F_1 \text{ and } r_2 \models F_2 \\
r \models F_1 \mathrel{-\!\!*} F_2 & \Leftrightarrow & \forall r'.\, r \circ r' \text{ defined and } r' \models F_1 \text{ implies } r \circ r' \models F_2
\end{array}
$$

Our contribution: classical BI *(*CBI*)*

- Why aren't there multiplicative versions of $\bot, \neg, \vee$?
- We obtain CBI by adding them to BBI:

$$\begin{array}{lcccccc} \textit{Additive:} & \top & \bot & \neg & \wedge & \vee & \rightarrow \\ \textit{Multiplicative:} & \top^* & \bot^* & \sim & * & \mathbin{\overset{*}{\vee}} & \mathrel{-\!\!*} \end{array}$$

and considering multiplicatives to behave classically.

Problems

- Does a logic like CBI even make any sense?

- How do we interpret the new connectives?
- Is there a nice proof theory?
- What are the potential applications?

Dualising resource models of CBI

- A CBI-model is given by a tuple $\langle R, \circ, e, -, \infty \rangle$, where:
  - $\langle R, \circ, e \rangle$ is a BBI-model;
  - $\infty \in R$ and $- : R \to R$;
  - for all $r \in R$, $-r$ is the unique solution to $r \circ -r = \infty$.
- Natural interpretation: models of dualising resources.
- Every Abelian group is a CBI-model (with $\infty = e$).
- We interpret $\bot^*, \sim, \mathrel{\overset{*}{\vee}}$ as follows:

$$
\begin{array}{rcl}
r \models \bot^* & \Leftrightarrow & r \neq \infty \\
r \models \sim F & \Leftrightarrow & -r \not\models F \\
r \models F_1 \mathrel{\overset{*}{\vee}} F_2 & \Leftrightarrow & r \models \sim(\sim F_1 * \sim F_2)
\end{array}
$$

Example: Personal finance

- Let $\langle \mathbb{Z}, +, 0, - \rangle$ be the Abelian group of integers (money):
- $m \models F$ means "£m is enough to make F true".
- Let C / W be the formulas *"I've enough money to buy cigarettes / whisky"*.

$m \models C * W \quad \Leftrightarrow$ *"£m is enough to buy both cigarettes and whisky"*

$m \models {\sim}C \quad \Leftrightarrow$ *"I owe less than the price of a pack of cigarettes"*

$m \models C \stackrel{*}{\triangledown} W \quad \Leftrightarrow$ *"so long as I don't spend more than the price of cigarettes, I can definitely still buy whisky"*

Proof theory

- We give a display calculus proof system, $\mathrm{DL}_{\mathrm{CBI}}$, for CBI.
- Display calculi are essentially generalised sequent calculi, with an enriched meta-level.
- Main technical results about $\mathrm{DL}_{\mathrm{CBI}}$:

Theorem (Cut-elimination)

Any $\mathrm{DL}_{\mathrm{CBI}}$ proof can be transformed into a cut-free proof.

Theorem (Soundness)

Any $\mathrm{DL}_{\mathrm{CBI}}$-derivable proof judgement is valid.

Theorem (Completeness)

Any valid proof judgement is $\mathrm{DL}_{\mathrm{CBI}}$-derivable.

Applications of CBI*: what cannot be done*

Proposition

CBI *is a non-conservative extension of* BBI*. That is, there are formulas of* BBI *that are valid wrt.* CBI *but not* BBI*.*

- Separation logic heap model does not extend to a CBI-model.
- Consequence: we cannot (directly) apply CBI reasoning principles such as $F \mathrel{-\!\!*} G \equiv {\sim}F \overset{*}{\vee} G$ to the heap model.
- Look for applications where resources are naturally dualising.

A CBI-*model of financial portfolios*

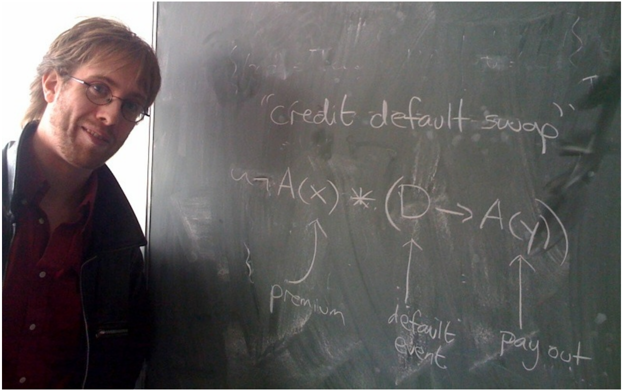
- Let ID be an infinite set of identifers.
- Let P be the set of portfolios: functions $p : ID \to \mathbb{Z}$ s.t. $p(x) \neq 0$ for only finitely many $x \in ID$.
- Define composition $+$, involution $-$ and empty portfolio e:

$$\begin{aligned}(p_1 + p_2)(x) &= p_1(x) + p_2(x) \\ (-p)(x) &= -p(x) \\ e(x) &= 0\end{aligned}$$

- $\langle P, +, e, - \rangle$ is an Abelian group, thus also a CBI-model.

Credit crunch solved!

Let $A(x)$ represent a portfolio consisting of asset x.
Then $\sim\neg A(x)$ represents a portfolio consisting of liability x.



Summary of CBI

Model theory: based on involutive commutative monoids

- multiplicatives are classical
- a non-conservative extension of BBI

Proof theory: a display calculus gives us:

- cut-elimination
- soundness
- completeness

Applications: reasoning about dualising resources, e.g.:

- money;
- permissions;
- bi-abduction.