

# *An Introduction to Cyclic Proofs*

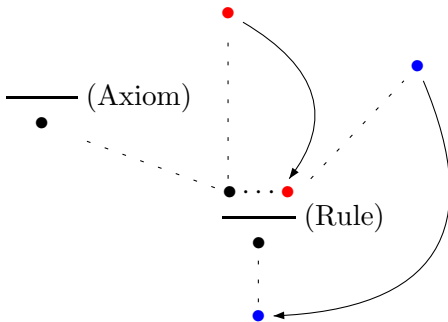
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PARIS workshop, FLoC, Oxford, 7th July 2018


## Cyclic pre-proofs

A **cyclic pre-proof** is a derivation tree with a **backlink** from each open leaf (“bud”) to an identical “**companion**”:

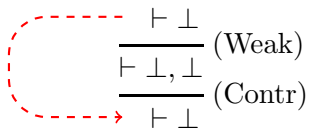


**Cyclic proof** = pre-proof  $\mathcal{P}$  + soundness condition  $S(\mathcal{P})$ .

## *An invalid pre-proof*

$$\begin{array}{c} \text{---} \vdash \perp \\ \hline \vdash \perp, \perp \text{ (Weak)} \\ \hline \vdash \perp \text{ (Contr)} \end{array}$$


## *An invalid pre-proof*



The diagram illustrates a circular dependency between two inference rules. At the top, the expression  $\vdash \perp$  is shown above a horizontal line, with the label "(Weak)" to its right. Below this line is the expression  $\vdash \perp, \perp$ , which is itself above another horizontal line, with the label "(Contr)" to its right. Below the second line is the expression  $\vdash \perp$ . A red dashed line starts from the left of the top  $\vdash \perp$ , curves down and left, then curves down and right to point at the bottom  $\vdash \perp$ , forming a loop that indicates the bottom  $\vdash \perp$  is derived from the top  $\vdash \perp$ .

$$\frac{\vdash \perp}{\vdash \perp, \perp} \text{ (Weak)}$$
$$\frac{\vdash \perp, \perp}{\vdash \perp} \text{ (Contr)}$$

- This is certainly a pre-proof, but obviously it cannot be accepted as valid!

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- This is certainly a pre-proof, but obviously it cannot be accepted as valid!
- Here, we formed a cycle but failed to make any appreciable “progress”.

## *The need for a soundness condition*

- In any reasonable proof system the rules must be **locally sound**: if all premises of the rule are valid then so is its conclusion.

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- However, when proofs are cyclic graphs, local soundness just says that if the root judgement is invalid then there is an **infinite path of invalid judgements** in the tree.



## *The need for a soundness condition*

- In any reasonable proof system the rules must be **locally sound**: if all premises of the rule are valid then so is its conclusion.
- When proofs are finite trees, this guarantees that any provable judgement is valid: supposing not, then some axiom in the tree must be invalid, contradiction.
- However, when proofs are cyclic graphs, local soundness just says that if the root judgement is invalid then there is an **infinite path of invalid judgements** in the tree.
- A soundness condition for cyclic proofs must therefore **rule out** the existence of such paths.

## *Infinite descent*

*Because the ordinary methods now in the books were insufficient for demonstrating such difficult propositions, I finally found a totally unique route for arriving at them ... which I called **infinite descent** ...*

*If there were any integral right triangle that had an area equal to a square, there would be another triangle **less than that one which would have the same property**...*

*Now it is the case that, given a number, **there are not infinitely many numbers less than that one in descending order** ... Whence one concludes that it is **therefore impossible** that there be any right triangle of which the area is a square...*

Pierre de Fermat, *Relation des nouvelles decouvertes en la science des nombres*, letter to Pierre de Carcavi, 1659

## *Infinite descent example*

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$\sqrt{2}$  is not rational.

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Suppose for contradiction that  $\sqrt{2} = x/y$  for  $x, y \in \mathbb{N}$ . Then  $x^2 = 2y^2$ . Consequently  $x(x - y) = y(2y - x)$ , so that:

$$\frac{2y - x}{x - y} = \frac{x}{y} = \sqrt{2}.$$

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Define  $x' = 2y - x$  and  $y' = x - y$ . Then  $x'/y' = \sqrt{2}$ .

Now observe that  $1 < x^2/y^2 < 4$ , so  $y < x < 2y$ , and so  $0 < y' < y$ .

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Define  $x' = 2y - x$  and  $y' = x - y$ . Then  $x'/y' = \sqrt{2}$ .

Now observe that  $1 < x^2/y^2 < 4$ , so  $y < x < 2y$ , and so  $0 < y' < y$ . But then we have  $x', y' \in \mathbb{N}$  such that  $\sqrt{2} = x'/y'$ , and  $y' < y$ . This gives an **infinite descent** from  $y$ . □

*Example:  $\mu$ -calculus properties of processes*

“Clock” process  $Cl$  repeatedly ticks:

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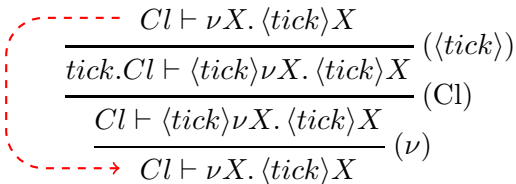
$$\frac{\frac{Cl \vdash \nu X. \langle \text{tick} \rangle X}{\text{tick}.Cl \vdash \langle \text{tick} \rangle \nu X. \langle \text{tick} \rangle X} (\langle \text{tick} \rangle)}}{\frac{Cl \vdash \langle \text{tick} \rangle \nu X. \langle \text{tick} \rangle X}{Cl \vdash \nu X. \langle \text{tick} \rangle X} (\nu)} (\text{Cl})$$

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1. By supposition there are no infinite *tick* sequences from  $Cl$ . However, the infinite path *does* create such an infinite sequence, since  $\langle tick \rangle$  is applied infinitely often.
2. There must be some ordinal-indexed **overapproximation** of the fixed point  $\nu^\alpha X. \langle tick \rangle X$  of which  $Cl$  is not a member. Unfolding  $\nu X$  infinitely often (by  $(\nu)$ ) creates an infinite descending chain of such ordinals, from  $\alpha$  — but these are well-founded.

## *Hoare logic*

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Then  $\{P\} C \{Q\}$  is **valid** when:

if  $\sigma \models P$  and  $\langle C, \sigma \rangle \rightarrow^* \langle \sigma' \rangle$  then  $\sigma' \models Q$  .

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while i>0 {if * then i--;};
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But program commands are **symbolically executed** infinitely often along this path. Thus the assumed execution from  $\langle C, \sigma \rangle$  is in fact **infinite**: contradiction.

## *A quick aside*

One can draw a rough analogy between cyclic Hoare proofs and **abstract interpretation**, also used to verify imperative code:

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<u>Abstract interpretation</u>		<u>Cyclic Hoare proofs</u>
--------------------------------	--	----------------------------

abstract domain	~	formula language
symbolic execution	~	symbolic execution
widening	~	generalisation
narrowing	~	instantiation
invariance	~	proof cycle

## *Inductive definitions in first-order logic*

Consider these **inductive definitions** of predicates  $N, E, O$ :

$$\begin{array}{ll} \Rightarrow N0 & \Rightarrow E0 \\ Nx \Rightarrow Nsx & Ex \Rightarrow Osx \\ Ox \Rightarrow Esx & \end{array}$$

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These definitions generate **case-split rules**, e.g., for  $N$ :

$$\frac{\Gamma, t = 0 \vdash \Delta \quad \Gamma, t = sx, Nx \vdash \Delta}{\Gamma, Nt \vdash \Delta} \text{ (Case } N\text{)}$$

(where  $x$  is fresh).

Note that  $Nx$  in the right-hand premise is obtained by *unfolding*  $Nt$  in the conclusion.

## *Example, inductive definitions*

We'll prove that every natural number is either even or odd, i.e.  
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$$\begin{array}{c}
 \frac{}{\vdash E0, O0} (E) \qquad \frac{Nx \vdash Ox, Ex}{Ny \vdash Oy, Ey} (\text{Subst}) \\
 \frac{}{x = 0 \vdash Ex, Ox} (=) \qquad \frac{Ny \vdash Oy, Ey}{Ny \vdash Oy, Osy} (O) \\
 \frac{}{x = 0 \vdash Ex, Ox} (=) \qquad \frac{Ny \vdash Oy, Osy}{Ny \vdash E sy, O sy} (E) \\
 \frac{}{x = 0 \vdash Ex, Ox} (=) \qquad \frac{}{x = sy, Ny \vdash Ex, Ox} (=) \\
 \hline
 \frac{}{x = 0 \vdash Ex, Ox} (=) \qquad \frac{}{x = sy, Ny \vdash Ex, Ox} (=) \qquad \text{(Case } N) \\
 \hline
 \frac{Nx \vdash Ex, Ox}{Nx \vdash Ex \vee Ox} (\vee)
 \end{array}$$

Note that here we examine formulas on the **left** of the turnstile!

## *Explanation of soundness*

Suppose that  $Nx \vdash Ex \vee Ox$  is invalid, meaning that  $M \models_{\rho} Nx$  (for some structure  $M$  and valuation  $\rho$ ) but  $M \not\models_{\rho} Ex \vee Ox$ .

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1. that  $\llbracket N \rrbracket_M$  is a well-founded set and we have an infinite descent in these “numerals”, from  $\rho(x)$ , because of the infinite unfolding of  $Nx$ ; or

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1. that  $\llbracket N \rrbracket_M$  is a well-founded set and we have an infinite descent in these “numerals”, from  $\rho(x)$ , because of the infinite unfolding of  $Nx$ ; or
2. that if  $\rho(x) \in \llbracket N \rrbracket_M$  that it is a member of some **underapproximation**  $\llbracket N \rrbracket_M^{\alpha}$ , and we have an infinite descent in these approximant ordinals, again because of the infinite unfolding of  $N$ .

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 \hline
 x = 0 \vdash Nx \\
 \hline
 \frac{}{Ex \vdash Nx} \\
 \hline
 \frac{}{x = sy, Oy \vdash Nx} (=) \\
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 \frac{}{x = sy, Ey \vdash Nx} (=) \\
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 \frac{}{Ex \vee Ox \vdash Nx} (\vee)
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$\frac{Ox \vdash Nx}{Oy \vdash Ny} \text{ (Subst)} \quad \frac{Oy \vdash Ny}{Oy \vdash Nsy} (N)$

$\frac{Ex \vdash Nx}{Ey \vdash Ny} \text{ (Subst)} \quad \frac{Ey \vdash Ny}{Ey \vdash Nsy} (N)$

$\frac{Oy \vdash Nsy}{x = sy, Oy \vdash Nx} (=) \quad \frac{Ey \vdash Nsy}{x = sy, Ey \vdash Nx} (=)$

$\frac{Oy \vdash Nsy}{x = sy, Oy \vdash Nx} \text{ (Case } E) \quad \frac{Ey \vdash Nsy}{x = sy, Ey \vdash Nx} \text{ (Case } O)$

$\frac{Ex \vdash Nx}{Ox \vdash Nx} \text{ (Case } E) \quad \frac{Ox \vdash Nx}{Ox \vdash Nx} \text{ (Case } O)$



## *Remark on soundness*

Our soundness justifications often rely on reasoning of the form “*this* formula instance in the proof is a fixed point unfolding of *that* one”.

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Some proof rules can complicate this reasoning. Some instances:

$$\frac{A \vdash B}{A, Px \vdash B} \text{ (Weak)} \quad \frac{Py \vdash B}{Px, x = y \vdash B} (=) \quad \frac{Px \vdash F \quad F \vdash B}{Px \vdash B} \text{ (Cut)}$$
$$\frac{Px \vdash Fx}{Pz \vdash Fz} \text{ (Subst)} \quad \frac{x = sy, Ey \vdash B}{Ox \vdash B} \text{ (Case } O\text{)}$$

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$$\frac{Px \vdash Fx}{Pz \vdash Fz} \text{ (Subst)} \quad \frac{x = sy, Ey \vdash B}{Ox \vdash B} \text{ (Case } O)$$

Dealing with this is essentially a matter of book-keeping. And it might not be necessary if there are no tricky rules.

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- A **trace** along a path in a pre-proof is obtained by chaining trace pairs together in the obvious way.
- A trace is **infinitely progressing** if it contains infinitely many progressing trace pairs.

## *A general soundness condition*

Given some appropriate<sup>1</sup> notion of “trace pairs” for a cyclic proof system, one can then state a general soundness condition:

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## *A general soundness condition*

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Virtually all the cyclic systems I know use a condition of this form, or which can be rewritten as such.

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## *Two relevant facts*

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1. **Cyclic proofs then become sound.** If not, then there is an infinite path of invalid judgements in the proof. There is an infinitely progressing trace following this path. This can be used to realise an infinite descending chain of values in a well-founded set: contradiction.

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1. **Cyclic proofs then become sound.** If not, then there is an infinite path of invalid judgements in the proof. There is an infinitely progressing trace following this path. This can be used to realise an infinite descending chain of values in a well-founded set: contradiction.
2. **It is decidable whether a pre-proof  $\mathcal{P}$  is a cyclic proof or not.**

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1. **Cyclic proofs then become sound.** If not, then there is an infinite path of invalid judgements in the proof. There is an infinitely progressing trace following this path. This can be used to realise an infinite descending chain of values in a well-founded set: contradiction.
2. **It is decidable whether a pre-proof  $\mathcal{P}$  is a cyclic proof or not.** Build two Büchi automata:  $B_1$  accepting all infinite paths in  $\mathcal{P}$ ; and  $B_2$  accepting all paths with an infinitely progressing trace on some tail. The soundness condition amounts to checking  $\mathcal{L}(B_1) \subseteq \mathcal{L}(B_2)$ .

## *Some logics with cyclic proof systems*

- $\mu$ -calculus (modal, first-order, process verification)
- temporal logic (CTL, LTL, . . .)
- first-order logic with ind. defns
- separation logic with ind. defns
- Hoare logic and variants (e.g. termination)
- linear logic with fixed points
- modal logic (of certain kinds)
- Kleene algebra
- combinations of the above

This is by no means a complete list!