Parametric completeness for separation theories (via hybrid logic)

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New York University, 11 December 2014

Joint work with Jules Villard

Part I

Introduction, motivation and background

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  - validity in a (class of) intended model(s) of the logic.

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- Here, we examine these questions in the context of pure separation logic, where
  - the language is given by the logic Boolean BI (BBI);
  - the intended models are given by separation theories, which specify a collection of useful model properties.

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3. We then propose an extension of BBI based on hybrid logic, which adds a theory of naming to BBI, and show that these properties become definable to this extension.
4. We give proof systems for our hybrid logic that is parametrically complete w.r.t. the axioms defining separation theories.

Part II

Boolean BI

BBI: *language and provability*

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$$A * B \vdash B * A \qquad A * (B * C) \vdash (A * B) * C$$

$$A \vdash A * \mathrm{I} \qquad A * \mathrm{I} \vdash A$$

$$\frac{A_1 \vdash B_1 \quad A_2 \vdash B_2}{A_1 * A_2 \vdash B_1 * B_2} \qquad \frac{A * B \vdash C}{A \vdash B \mathrel{-\!\!*} C} \qquad \frac{A \vdash B \mathrel{-\!\!*} C}{A * B \vdash C}$$

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- e is the empty heap that is undefined everywhere.

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Semantics of formula A wrt. BBI-model $M = \langle W, \circ, E \rangle$, valuation ρ, and $w \in W$ given by relation $M, w \models_\rho A$:

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Theorem (Galmiche and Larchey-Wendling 2006)

Provability in BBI *coincides with validity in* BBI*-models.*

Part III

(Un)definable properties in BBI

Separation theories

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Cross-split property: whenever $(a \circ b) \cap (c \circ d) \neq \emptyset$, there exist ac, ad, bc, bd such that $a \in ac \circ ad$, $b \in bc \circ bd$, $c \in ac \circ bc$ and $d \in ad \circ bd$.

Definable properties

A property $\mathcal{P}$ of BBI-models is said to be $\mathcal{L}$-definable if there exists an $\mathcal{L}$-formula A such that for all BBI-models M,

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Proof.

Just directly verify the needed biimplication. □

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Proof.

Structural induction on A. □

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Lemma

Let $\mathcal{P}$ *be a property of* BBI-*models, and suppose that there exist* BBI-*models* M_1 *and* M_2 *such that* $M_1, M_2 \in \mathcal{P}$ *but* $M_1 \uplus M_2 \notin \mathcal{P}$. *Then* $\mathcal{P}$ *is not* BBI-*definable.*

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The disjoint union of any two single-unit BBI-models (e.g. two copies of $\mathbb{N}$ under addition) is not a single-unit model, so we are done by the above Lemma. □

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Proof.

E.g., for functionality, we build models M and M' such that there is a bounded morphism from M to M', but M is functional while M' is not. See paper for details. □

Part IV

Hybrid extensions of BBI

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Easy to see that HyBBI is a conservative extension of BBI.

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Proof.

Easy verifications! □

A word about cross-split

We have brushed over the cross-split property:

$(a \circ b) \cap (c \circ d) \neq \emptyset$, *implies* $\exists ac, ad, bc, bd$ *with*
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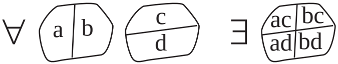

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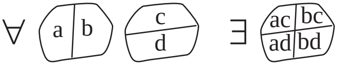

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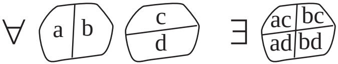

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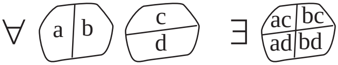

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then cross-split is definable as the pure formula

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Part V

Parametric completeness for HyBBI(↓)

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$$\frac{@_\ell(k * k') \wedge @_k A \wedge @_{k'} B \vdash C}{@_\ell(A * B) \vdash C} \quad k, k' \text{ not in } A,\ B,\ C \text{ or } \{\ell\} \quad \text{(Paste } *)$$

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Proposition (Soundness)

Any $\mathbf{K}_{\mathrm{HyBBI}(\downarrow)}$*-provable sequent is valid in all* BBI*-models.*

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(In our case, we also have to show that the canonical model is really a BBI-model.)

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- By the above Lemma + MCS properties, the Ax are valid in M.
- That is, $\mathbf{K}_{\mathrm{HyBBI}(\downarrow)} + \mathrm{Ax}$ is complete for the models s.t. Ax!

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Theorem (Parametric completeness)

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Corollary

By a suitable choice of axioms, we have a sound and complete axiomatic proof system for any given separation theory from our collection.

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Thanks for listening!

Prelim version of paper available from authors' webpages:

J. Brotherston and J. Villard.
Parametric completeness for separation theories.
To appear at *POPL'14*.