# *A unified display proof theory for bunched logic*

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- Relevant logic replaces some of the standard 'additive' connectives, which obey weakening and contraction, with 'multiplicative' variants which do not;
- Bunched logic is like relevant logic, but retains the additive connectives which relevant logic throws away on philosophical grounds (e.g. $\rightarrow$).

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- This gives a nice Kripke-style resource semantics: Additive connectives have their usual meaning, and multiplicatives denote resource composition properties:

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\begin{aligned}
r \models F_1 \wedge F_2 &\Leftrightarrow r \models F_1 \text{ and } r \models F_2 \\
r \models F_1 * F_2 &\Leftrightarrow r = r_1 \circ r_2 \text{ and } r_1 \models F_1 \text{ and } r_2 \models F_2
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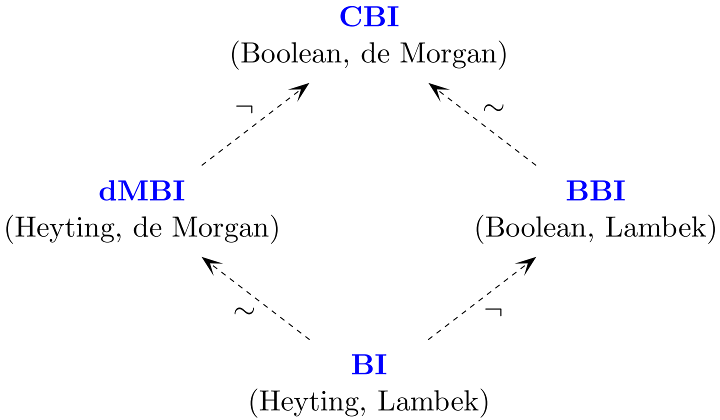
  (where $\circ$ is a binary monoid operation).
- Taking particular models gives us separation logic and other spatial logics (used in program verification).

# *The bunched logic family*

Additives / multiplicatives can be classical or intuitionistic:




- Subtitles (X,Y) indicate the underlying algebras.
- Arrows denote addition of classical negations $\neg$ or $\sim$.

# *Bunched logics via elementary logics*

$$\begin{array}{lcccccc} \textit{Additives:} & \top & \bot & \neg & \vee & \wedge & \rightarrow \\ \textit{Multiplicatives:} & \top^* & \bot^* & \sim & \overset{*}{\vee} & * & -\!\!* \end{array}$$

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- IL and CL are standard intuitionistic / classical logic over the additives;
- LM and dMM are (commutative and associative) Lambek / de Morgan logic over the multiplicatives;
- Define:

$$\begin{aligned} \mathrm{BI} &= \mathrm{IL} + \mathrm{LM} \\ \mathrm{BBI} &= \mathrm{CL} + \mathrm{LM} \\ \mathrm{dMBI} &= \mathrm{IL} + \mathrm{dMM} \\ \mathrm{CBI} &= \mathrm{CL} + \mathrm{dMM} \end{aligned}$$

  where + is union of minimal proof systems for the logics.

# LBI: *the* BI *sequent calculus*

- Sequents are $\Gamma \vdash F$ where $F$ a formula and $\Gamma$ a bunch:

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- Rules for $-\!\!*$ are:

$$\frac{\Delta \vdash F_1 \quad \Gamma(F_2) \vdash F}{\Gamma(\Delta\,,\,F_1 -\!\!* F_2) \vdash F}\,(-\!\!*\mathrm{L}) \qquad \frac{\Gamma\,,\,F \vdash G}{\Gamma \vdash F -\!\!* G}\,(-\!\!*\mathrm{R})$$

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  where $\Gamma(\Delta)$ is bunch $\Gamma$ with sub-bunch $\Delta$;

- LBI satisfies cut-elimination (Pym '02).
- Unfortunately cut-elimination breaks if we try to extend LBI to BBI, dMBI, CBI in the obvious way.

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- In display calculi, one can rearrange consecutions:

*Definition*

$\equiv_D$ is a display-equivalence if for any antecedent (consequent) part $Z$ of $X \vdash Y$ we have $X \vdash Y \equiv_D Z \vdash W \quad (W \vdash Z)$.

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- Belnap '82 gives a set of syntactic conditions for display calculi which guarantee cut-elimination.

# *Display calculus: syntax*

- Structures are constructed from formulas and structural connectives:

| *Additive* | *Multiplicative* | *Arity* | *Antecedent* | *Consequent* |
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| $\emptyset$ | $\varnothing$ | 0 | truth | falsity |
| $\sharp$ | $\flat$ | 1 | negation | negation |
| ; | , | 2 | conjunction | disjunction |
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- Antecedent / consequent parts of consecutions $X \vdash Y$ are similar to positive / negative occurrences in formulas.
- We give display calculi for IL, CL, LM and dMM. Form of antecedent and consequent parts is restricted in each case.

# $\mathrm{DL_{CL}}$: *a display calculus for* CL

**Antecedent connectives:** $\emptyset \quad \sharp \quad ;$

**Consequent connectives:** $\emptyset \quad \sharp \quad ;$

**Display postulates:**

$$\begin{array}{ccccc} X ; Y \vdash Z & <>_D & X \vdash \sharp Y ; Z & <>_D & Y ; X \vdash Z \\ X \vdash Y ; Z & <>_D & X ; \sharp Y \vdash Z & <>_D & X \vdash Z ; Y \\ X \vdash Y & <>_D & \sharp Y \vdash \sharp X & <>_D & \sharp\sharp X \vdash Y \end{array}$$

**Logical rules:**

$$\frac{F \vdash X \quad G \vdash X}{F \vee G \vdash X}\,(\vee\mathrm{L}) \qquad \frac{X \vdash F_1 ; F_2}{X \vdash F_1 \vee F_2}\,(\vee\mathrm{R}) \qquad \text{(etc.)}$$

**Structural rules:**

$$\frac{\emptyset ; X \vdash Y}{X \vdash Y}\,(\emptyset\mathrm{L}) \qquad \frac{X \vdash Z}{X ; Y \vdash Z}\,(\mathrm{WkL}) \qquad \text{(etc.)}$$

## $\mathrm{DL_{LM}}$: *a display calculus for* LM

**Antecedent connectives:** $\varnothing \quad ,$

**Consequent connectives:** $\multimap$

**Display postulates:** $X \,,\, Y \vdash Z \;\; <>_D \;\; X \vdash Y \multimap Z \;\; <>_D \;\; Y \,,\, X \vdash Z$

**Logical rules:**

$$\frac{X \vdash F \quad G \vdash Y}{F \mathrel{-\!\!*} G \vdash X \multimap Y}\,(\mathrel{-\!\!*}\mathrm{L}) \qquad \frac{X \vdash F \multimap G}{X \vdash F \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}\mathrm{R}) \qquad \text{(etc.)}$$

**Structural rules:**

$$\frac{\varnothing \,,\, X \vdash Y}{X \vdash Y}\,(\varnothing\mathrm{L}) \qquad \frac{W \,,\, (X \,,\, Y) \vdash Z}{(W \,,\, X) \,,\, Y \vdash Z}\,(\mathrm{MAL})$$

## *Display calculi for bunched logics*

We obtain display calculi $\mathrm{DL}_{\mathcal{L}}$ for $\mathcal{L} \in \{\mathrm{BI}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$ by:

$$\mathrm{DL}_{\mathcal{L}_1+\mathcal{L}_2} = \mathrm{DL}_{\mathcal{L}_1} + \mathrm{DL}_{\mathcal{L}_2}$$

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*$X \vdash Y$ is $\mathrm{DL}_{\mathcal{L}}$-provable iff its formula translation is provable in the minimal proof system for $\mathcal{L}$.*

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*$X \vdash Y$ is $\mathrm{DL}_{\mathcal{L}}$-provable iff its formula translation is provable in the minimal proof system for $\mathcal{L}$.*

*Theorem (Cut-elimination)*

*Any $\mathrm{DL}_{\mathcal{L}}$ proof of $X \vdash Y$ can be algorithmically transformed into a cut-free $\mathrm{DL}_{\mathcal{L}}$ proof of $X \vdash Y$.*

## *Translating* LBI *into* $\mathrm{DL_{BI}}$

Recall the LBI rules for $\multimap$:

$$\frac{\Delta \vdash F_1 \quad \Gamma(F_2) \vdash F}{\Gamma(\Delta\ ,\ F_1 \multimap F_2) \vdash F}\,(\multimap\mathrm{L}) \qquad\qquad \frac{\Gamma\ ,\ F \vdash G}{\Gamma \vdash F \multimap G}\,(\multimap\mathrm{R})$$

$(\multimap\mathrm{R})$ has a direct equivalent in $\mathrm{DL_{BI}}$, while $(\multimap\mathrm{L})$ can be derived in $\mathrm{DL_{BI}}$ as follows:

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$$\frac{\dfrac{\Delta \vdash F_1 \qquad \dfrac{\Gamma(F_2) \vdash F}{F_2 \vdash X}\,(\mathrm{D}\equiv)}{\Delta \,,\, F_1 -\!\!* F_2 \vdash X}\,(-\!\!*\mathrm{L})}{\Gamma(\Delta \,,\, F_1 -\!\!* F_2) \vdash F}\,(\mathrm{D}\equiv)$$

Translation preserves cut-freeness of proofs.

# *Translating* $\mathrm{DL_{BI}}$ *into* LBI

For any $\mathrm{DL_{BI}}$ consecution $X \vdash Y$ define $\ulcorner X \vdash Y \urcorner$ as the result of maximally applying transformations:

$$\begin{array}{rcl} X \vdash Y \Rightarrow Z & \mapsto & X \,;\, Y \vdash Z \\ X \vdash Y \multimap Z & \mapsto & X \,,\, Y \vdash Z \end{array}$$

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Then the rules of $\mathrm{DL_{BI}}$ are LBI-derivable under $\ulcorner - \urcorner$, e.g.:

$$\frac{\ulcorner X \vdash F \urcorner \quad \ulcorner G \vdash Y \urcorner}{\ulcorner X \,,\, F \mathrel{-\!\!*} G \vdash Y \urcorner} \quad = \quad \frac{X \vdash F \quad \Gamma(G) \vdash H}{\Gamma(X \,,\, F \mathrel{-\!\!*} G) \vdash H}$$

Translation again preserves cut-freeness of proofs.

# *Display calculi vs. sequent calculi*

- By the two previous translations we have:

*Proposition*

*There is a one-to-many correspondence between cut-free proofs in* LBI *and cut-free proofs in* $\mathrm{DL}_{\mathrm{BI}}$.

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- Thus we claim that our display calculi really are canonical proof systems for the bunched logics.

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- Cut-elimination provides structure and removes infinite branching points from the proof search space.
- Our calculi could be potentially be used in interactive theorem proving tools (proof-by-pointing) or to define partial search strategies.