Model Checking for Symbolic-Heap Separation Logic with Inductive Predicates

James Brotherston\textsuperscript{1} \hspace{1cm} Nikos Gorogiannis\textsuperscript{2} \hspace{1cm} Max Kanovich\textsuperscript{1} \\
Reuben Rowe\textsuperscript{1}

\textsuperscript{1}UCL \\
\textsuperscript{2}Middlesex University

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- In computer science, $S$ is typically a Kripke structure representing a system or program, and $A$ a formula of modal or temporal logic.

- More generally, $S$ could be any kind of mathematical structure and $A$ a formula describing such structures.
Model checking, in particular

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- We focus on the popular symbolic-heap fragment of separation logic, allowing arbitrary inductive predicates.
Symbolic-heap separation logic

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Pure formulas $\pi$ and spatial formulas $F$ given by:

$$\pi ::= t = t \mid t \neq t \quad F ::= \text{emp} \mid x \mapsto t \mid Pt \mid F \ast F$$

(where $P$ a predicate symbol, $t$ a tuple of terms).
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- $\mapsto$ (“points-to”) denotes an individual pointer to a record in the heap.

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- Symbolic heaps $A$ given by $\exists x. \Pi : F$, for $\Pi$ a set of pure formulas.
Inductive definitions in separation logic

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  \[ A \Rightarrow Pt \]

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- E.g., linked list segments with root \( x \) and tail element \( y \):

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\text{emp} \Rightarrow \text{ls } x x \\
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\end{align*}
\]

- E.g., **binary trees** with root \( x \) given by:

\[
\begin{align*}
x = \text{nil} : \text{emp} & \Rightarrow \text{bt} \ x \\
x \neq \text{nil} : x \mapsto (y, z) \ast \text{bt} \ y \ast \text{bt} \ z & \Rightarrow \text{bt} \ x
\end{align*}
\]
**Semantics**

- Models are **stacks** $s : \text{Var} \rightarrow \text{Val}$ paired with **heaps** $h : \text{Loc} \rightarrow_{\text{fin}} \text{Val}$. $\circ$ is union of **domain-disjoint** heaps; $e$ is the **empty** heap; nil is a **non-allocable** value.
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- Forcing relation $s, h \vdash A$ given by

  $$s, h \vdash \Phi t_1 = (\neq) t_2 \iff s(t_1) = (\neq) s(t_2)$$
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    s, h \models_\Phi t_1 = (\neq) t_2 & \iff s(t_1) = (\neq) s(t_2) \\
    s, h \models_\Phi \text{emp} & \iff h = e
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  s, h \models_\Phi P_i t \iff (s(t), h) \in [P_i]_\Phi
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  $s, h \models \Phi P_i t \iff (s(t), h) \in [P_i]^{\Phi}$
  $s, h \models \Phi F_1 \ast F_2 \iff \exists h_1, h_2. \ h = h_1 \circ h_2 \text{ and } s, h_1 \models \Phi F_1$
  and $s, h_2 \models \Phi F_2$
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    s, h \models_{\Phi} P_i t & \iff (s(t), h) \in [P_i]_{\Phi} \\
    s, h \models_{\Phi} F_1 \ast F_2 & \iff \exists h_1, h_2. h = h_1 \circ h_2 \text{ and } s, h_1 \models_{\Phi} F_1 \\
    & \text{ and } s, h_2 \models_{\Phi} F_2 \\
    s, h \models_{\Phi} \exists z. \Pi : F & \iff \exists v \in \text{Val}^{\mid z \mid}. s[z \mapsto v], h \models_{\Phi} \pi \text{ for all } \\
    & \pi \in \Pi \text{ and } s[z \mapsto v], h \models_{\Phi} F
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Given inductive rule set $\Phi$, the semantics $\llbracket P \rrbracket^\Phi$ of inductive predicate $P$ is the least fixed point of a monotone operator constructed from $\Phi$. 

E.g., recall linked list segments $\text{ls}$:

- $\text{emp} \Rightarrow \text{ls}$
- $x \not\equal{} \text{nil}$:
  - $x \mapsto \text{ls}$

The corresponding operator is:

$$\phi(X) = \{ (s, h) \mid s, h | = x = y \text{ and } s, h | = \text{emp}, \text{ or } s, h | = x \mapsto \text{ls} \}$$

where $X_{zy}$ is interpreted as $(z, y) \in X$. 

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Model checking problem (MC). Given an inductive rule set $\Phi$, stack $s$, heap $h$ and symbolic heap $A$, decide whether $s, h \models \Phi A$. 
**Problem statement**

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First, we can simplify the problem:

**Restricted model checking problem** (RMC). Given an inductive rule set $\Phi$, tuple of values $a$, heap $h$ and predicate symbol $P$, decide whether $(a, h) \in \lbrack P \rbrack_\Phi$. 
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Restricted model checking problem (RMC). Given an inductive rule set $\Phi$, tuple of values $a$, heap $h$ and predicate symbol $P$, decide whether $(a, h) \in \llbracket P \rrbracket_\Phi$.

Proposition
MC and RMC are (polynomially) equivalent.
Naive idea: apply inductive rules \textbf{backwards} to $Px$ until we reach the empty heap.
Subtle problem 1

Naive idea: apply inductive rules backwards to $Px$ until we reach the empty heap.

But, suppose $((a, b), h) \in \llbracket P \rrbracket^\Phi$, and is generated by the rule

$$\exists z. Pxz \ast Pzy \Rightarrow Pxy.$$ 

So, for some $c \in \text{Val}$, we have both $((a, c), h_1) \in \llbracket P \rrbracket^\Phi$ and $((c, b), h_2) \in \llbracket P \rrbracket^\Phi$, where $h = h_1 \circ h_2$. 
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But we do not know that $h_1$, $h_2$ are smaller than $h$.

**Moral:** compute “sub-models” of $(a, h)$ bottom-up until we reach a fixed point.
Subtle problem 2

Suppose \((a, e) \in \llbracket P \rrbracket^\Phi\) is generated by the rule

\[ \exists z. \ z \neq x : Qxz \Rightarrow Px. \]

So, for some \(b \in \text{Val}\), we have \(((a, b), e) \in \llbracket Q \rrbracket^\Phi\), where \(b \neq a\) and \(b\) (trivially) does not appear in the empty heap \(e\).
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Thus we must allow our sub-models to mention fresh, or “spare”, values not mentioned in \(a\) or \(h\).
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Fortunately, for any given set of definitions \(\Phi\), we can get away with using only finitely many of these spare values.
Our model checking constructions

Given $\Phi$, $a$ and $h$, define

$$\text{Good}(a, h) = a \cup \{nil\} \cup \text{all values in } h.$$
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1. we only consider heaps $h' \subseteq h$, and
2. all values instantiating variables must be taken from $\text{Good}(a, h) \cup \text{Spare}_{\Phi}(a, h)$. 
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Now let \( \beta \) be the maximum number of variables in any rule in \( \Phi \), and define \( \text{Spare}_\Phi(a, h) \) to be a set of \( \beta \) fresh values.

Then, given \( \Phi \), values \( a \) and heap \( h \) we define a monotone operator, similar to the one that constructs the semantics of inductive predicates except that

- we only consider heaps \( h' \subseteq h \), and
- all values instantiating variables must be taken from \( \text{Good}(a, h) \cup \text{Spare}_\Phi(a, h) \).

Again, we take the least fixed point of the operator, and write \( MC^\Phi_i(a, h) \) for the component corresponding to \( i \)th predicate.
Correctness

Lemma
For each predicate \( P_i \),

\[(a, h) \in \llbracket P_i \rrbracket^\Phi \iff (a, h) \in MC_i^\Phi(a, h) .\]
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Soundness ($\iff$) is easy — $MC_i^\Phi(a, h)$ only constructs models of $P_i$ by construction.
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Soundness ($\Leftarrow$) is easy — $MC_i^\Phi(a, h)$ only constructs models of $P_i$ by construction.

However, completeness ($\Rightarrow$) is hard: we have to show that $(a, h)$ must eventually turn up in $MC_i^\Phi(a, h)$, even if its derivation involves values outside $\text{Good}(a, h) \cup \text{Spare}_\Phi(a, h)$. Argument involves considering certain value substitutions and recycling values at each iteration of the fixed point construction.
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Proof.
It suffices to show that RMC is decidable: does \((a, h) \in [P_i]^{\Phi}\)?
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It suffices to show that $RMC$ is decidable: does $(a, h) \in [P_i]^{\Phi}$?

By our correctness lemma, this is equivalent to deciding whether $(a, h) \in MC_i^{\Phi}(a, h)$. 
Decidability

**Theorem**
The model checking problem $\text{MC}$ is decidable.

**Proof.**
It suffices to show that $\text{RMC}$ is decidable: does $(a, h) \in [P_i]^{\Phi}$?

By our correctness lemma, this is equivalent to deciding whether $(a, h) \in MC_i^{\Phi}(a, h)$.

But clearly $MC_i^{\Phi}(a, h)$ is a finite and computable set (because we restrict to subheaps of $h$ and a finite set of values), so this is a decidable problem. $\square$
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MC is EXPTIME-complete.
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**Proof.** Computing $MC_i^\Phi(a, h)$ decides the problem and can be seen to run in exponential time in the size of $(a, h, \Phi)$. 
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**Proposition**

When $\Phi$ and $a$ are fixed, MC is still NP-hard in the size of $h$. 
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**Proposition**
When \( \Phi \) and \( a \) are fixed, MC is still NP-hard in the size of \( h \).

**Proof.** By reduction from the triangle partition problem: given a graph \( G = (V, E) \) with \(|V| = 3q\) for some \( q > 0\), decide whether there is a partition of \( G \) into triangles.
MEM: Restriction to memory-consuming rules

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or

$$\exists z. \Pi : F \ast x \mapsto t \Rightarrow Px.$$ 

i.e., one or more pointers are “consumed” when recursing.
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i.e., one or more pointers are “consumed” when recursing.

In practice, almost all predicate definitions in the literature fall into MEM.
Model checking in the MEM fragment

Theorem
MC ∈ NP when all predicates are restricted to MEM.
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*Proof*. Given predicate $P_i$, values $a$ and heap $h$, we can search backwards by applying inductive rules to $(a, h) \in [P_i]$, noting that we can confine the search space of values using our previous observations. This search must terminate because at least one heap cell is consumed with each recursion.
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Proof. Given predicate $P_i$, values $a$ and heap $h$, we can search backwards by applying inductive rules to $(a, h) \in \lbrack P_i \rbrack$, noting that we can confine the search space of values using our previous observations. This search must terminate because at least one heap cell is consumed with each recursion.

Theorem
MC is in fact NP-hard for MEM (thus NP-complete), even when some further restrictions are added.
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E.g., consider two list definitions

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\begin{align*}
x &= y : \text{emp} & \Rightarrow & \text{ls}(x, y) \\
\exists z. \ x \mapsto z \ast \text{ls}(z, y) & \Rightarrow & \text{ls}(x, y)
\end{align*}
\]

\[
\begin{align*}
x &= y : \text{emp} & \Rightarrow & \text{rls}(x, y) \\
\exists z. \ x \neq y : & \text{rls}(x, z) \ast z \mapsto y & \Rightarrow & \text{rls}(x, y)
\end{align*}
\]

The existential \( z \) is constructively valued in \( \text{ls} \), but not in \( \text{rls} \).
A predicate $P_i$ is said to be deterministic (in an inductive rule set $\Phi$) if for any two of its inductive rules and any stack, the stack can satisfy the pure part of at most one of the rules.
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Again, take the list definitions:

$$x = y : \text{emp} \Rightarrow ls(x, y)$$
$$\exists z. x \mapsto z \ast ls(z, y) \Rightarrow ls(x, y)$$

$$x = y : \text{emp} \Rightarrow rls(x, y)$$
$$\exists z. x \neq y : rls(x, z) \ast z \mapsto y \Rightarrow rls(x, y)$$

Here, $rls$ is deterministic, but $ls$ is not.
Results on CV + DET fragments

Theorem

MC is PTIME-solvable when all predicates are in MEM + CV + DET.
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Proof.
Like in the MEM case, we can search backwards for a derivation of \((a, h) \in \mathcal{F}[P_i]\) using inductive rules. MEM ensures termination. DET ensures at most one inductive rule can apply, and CV ensures it can be instantiated in only one way.
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**Proof.**

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**Theorem**

If we remove any of the restrictions MEM, CV, DET, then the complexity of MC becomes PSPACE-hard or worse!
### Summary of problem complexities

<table>
<thead>
<tr>
<th>non-MEM</th>
<th>EXPTIME</th>
<th>EXPTIME</th>
<th>EXPTIME</th>
<th>(\geq) PSPACE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEM</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
<td>PTIME</td>
</tr>
</tbody>
</table>

### Time Complexity

<table>
<thead>
<tr>
<th>CV</th>
<th>DET</th>
<th>(CV + DET)</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-MEM</td>
<td>EXPTIME</td>
<td>EXPTIME</td>
</tr>
<tr>
<td>MEM</td>
<td>NP</td>
<td>NP</td>
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Implementation

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Tested on a range of annotated test programs, falling into various fragments, taken from the Verifast tool (Jacobs et al., Leuven).

Average-case performance is in line with predicted complexity bounds.

Thus, run-time verification is broadly practical for predicates in MEM + CV + DET; more complicated predicates can play a role in unit testing.
Conclusions

• Main contribution: for symbolic-heap separation logic with arbitrary inductive predicates, the model checking problem is decidable and indeed EXPTIME-complete.
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Conclusions

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• However, in practice most predicates are memory-consuming, i.e. in MEM, in which case the problem becomes NP-complete.

• If we additionally insist on constructively valued (CV) and deterministic (DET) definitions (some are, some aren’t), then the problem becomes PTIME-solvable.
Future work

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• Investigate the complexity when we add classical conjunction $\land$ to the logic? (Satisfiability becomes undecidable.)

• Investigate complexity of satisfiability for combinations of $\text{MEM/CV/DET}$. 

• Implementing the $\text{NP}$ algorithm for the $\text{MEM}$ fragment can be expected to yield better implementation performance (on $\text{MEM}$).

• Disprove entailments using model checking?
Thanks for listening!

Try our techniques within the Cyclist distribution:

github.com/ngorogiannis/cyclist

Also available as an official POPL’16 Artefact.