Proof theory for Boolean bunched logic

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Gentzen-style proof systems

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where Γ, Δ are sets, multisets or even more exotic structures.

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Characteristic feature: for any logical connective there should be proof rules explaining how to introduce that connective on the left and right of the conclusion of the rule.

There are also structural rules that only involve sequent structure, not logical connectives.

Example: Gentzen's LK

E.g., in Gentzen's LK for classical propositional logic, the sequents are built from sets, interpreted as

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and the rules for $\rightarrow$ are:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} (\rightarrow\text{L}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} (\rightarrow\text{R})$$

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Structural rules include:

$$\frac{\Gamma, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,(\mathrm{ContrL}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Delta'}\,(\mathrm{WkL})$$

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Getting rid of this is called cut-elimination, and proof theorists are absolutely obsessed with it!

BBI, *proof-theoretically*

Recall:

Provability in BBI is given by extending a Hilbert system for propositional classical logic by

$$A * B \vdash B * A \qquad A * (B * C) \vdash (A * B) * C$$

$$A \vdash A * \mathrm{I} \qquad A * \mathrm{I} \vdash A$$

$$\frac{A_1 \vdash B_1 \quad A_2 \vdash B_2}{A_1 * A_2 \vdash B_1 * B_2} \qquad \frac{A * B \vdash C}{A \vdash B \mathrel{-\!\!*} C} \qquad \frac{A \vdash B \mathrel{-\!\!*} C}{A * B \vdash C}$$

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- For quite a long time in the 2000s, researchers tried to find a nice sequent calculus for BBI, but cut-elimination typically failed.
- But we can give an analytic Gentzen system based on the slightly more general notion of display calculus.

Display calculus: an overview

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- Like sequent calculi, display calculi work with sequents of the form $X \vdash Y$, with left- and right-introduction rules for each logical connective.
- But, the structures X and Y can be structurally more complex than simple sets or multisets.
- Most importantly, display calculi allow us to rearrange sequents to focus on any individual part (like rearranging an equation in standard algebra).

Structures and interpretation

Structures X defined as follows:

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(N.B. (1) we switch from one interpretation function to the other when going inside $\sharp$; (2) $\varnothing$ is not allowed to occur "positively" in a sequent.)

Display property

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We call the reflexive-transitive closure of these rules display equivalence, $\equiv_D$. Then we get the crucial display property:

Theorem

For any "negative" part Z of $X \vdash Y$ we have $X \vdash Y \equiv_D Z \vdash W$, and for any "positive" part Z of $X \vdash Y$ we have $X \vdash Y \equiv_D W \vdash Z$.

Identity and logical rules

Identity rules:

$$\frac{}{A \vdash A}\,(\text{Id}) \qquad \frac{W \vdash Z}{X \vdash Y}\, W \vdash Z \equiv_D X \vdash Y (\equiv_D) \qquad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\,(\text{Cut})$$

Logical rules:

$$\frac{A \vdash X \quad B \vdash X}{A \vee B \vdash X}\,(\vee\text{L}) \qquad \frac{X \vdash A \quad B \vdash Y}{A \to B \vdash \sharp X \;;\; Y}\,(\to\text{L}) \qquad \frac{X \vdash A \quad B \vdash Y}{A \mathrel{-\!\!*} B \vdash X \;,\; Y}\,(\mathrel{-\!\!*}\text{L})$$

$$\frac{X \vdash A_1 \;;\; A_2}{X \vdash A_1 \vee A_2}\,(\vee\text{R}) \qquad \frac{X \;;\; A \vdash B}{X \vdash A \to B}\,(\to\text{R}) \qquad \frac{X \vdash A \;,\; B}{X \vdash A \mathrel{-\!\!*} B}\,(\mathrel{-\!\!*}\text{R})$$

(etc.)

Structural rules

$$\frac{X \,;\, X \vdash Z}{X \vdash Z}\,(\text{Contr}) \qquad \frac{X \vdash Z}{X \,;\, Y \vdash Z}\,(\text{Weak})$$

$$\frac{X \vdash Y}{\varnothing \,,\, X \vdash Y}\,(\varnothing 1) \qquad \frac{\varnothing \,,\, X \vdash Y}{X \vdash Y}\,(\varnothing 2) \qquad \frac{W \,,\, (X \,,\, Y) \vdash Z}{(W \,,\, X) \,,\, Y \vdash Z}\,(\text{Assoc})$$

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This can be done by appealing to the semantics, or by deriving in the Hilbert system for BBI.

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Lemma (2)

If $F \vdash G$ is provable in the Hilbert system for BBI *then it is provable in the display calculus too.*

Proof of completeness

Suppose $X \vdash Y$ is valid, i.e. $\Psi_X \models \Upsilon_Y$.

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\dfrac{\begin{array}{c}\text{(Lemma 1)}\\ \vdots \\ X \vdash \Psi_X\end{array} \qquad \dfrac{\begin{array}{c}\text{(Lemma 2)}\\ \vdots \\ \Psi_X \vdash \Upsilon_Y\end{array} \qquad \begin{array}{c}\text{(Lemma 1)}\\ \vdots \\ \Upsilon_Y \vdash Y\end{array}}{\Psi_X \vdash Y}\ \text{(Cut)}}{X \vdash Y}\ \text{(Cut)}
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Theorem (Cut-elimination)

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Belnap '82 famously gave a set of syntactic conditions C1–C8 on the proof rules of a display calculus which are sufficient to guarantee this.

Most are boring and easy to check. The only non-trivial one is that so-called principal cuts can be reduced to cuts on smaller formulas.

Principal cuts

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E.g., the following is a principal cut:

$$\dfrac{\dfrac{X \vdash F \,, G}{X \vdash F \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}\mathrm{R}) \quad \dfrac{Y \vdash F \quad G \vdash Z}{F \mathrel{-\!\!*} G \vdash Y \,, Z}\,(\mathrel{-\!\!*}\mathrm{L})}{X \vdash Y \,, Z}\,(\mathrm{Cut})$$

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Belnap's condition C8 requires us to show that we can transform this derivation into one where only cuts on the smaller subformulas, F and G, are used.

Cut elimination

Here's the reduced principal cut:

$$\dfrac{Y \vdash F \qquad \dfrac{\dfrac{\dfrac{X \vdash F \,,\, G}{X, F \vdash G}\,(\mathrm{D}\equiv) \qquad G \vdash Z}{X, F \vdash Z}\,(\mathrm{Cut})}{F \vdash X \,,\, Z}\,(\mathrm{D}\equiv)}{\dfrac{Y \vdash X \,,\, Z}{X \vdash Y \,,\, Z}\,(\mathrm{D}\equiv)}\,(\mathrm{Cut})$$

Other types of principal cut can be treated similarly. This gives us cut-elimination by Belnap's theorem.

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$$\text{cut-elimination} \not\Rightarrow \text{(semi)decidability}$$

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- In general, for both display and sequent calculi:

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- Indeed, as we shall see in the next lecture, BBI is still in fact undecidable.
- Cut-elimination provides structure and removes infinite branching points from the proof search space.

Further reading

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