Proof theory for Boolean bunched logic

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Gentzen-style proof systems

Gentzen-style systems are built around proof rules manipulating judgements called sequents, of the form:

\[ \Gamma \vdash \Delta \]

where \( \Gamma, \Delta \) are sets, multisets or even more exotic structures.
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Characteristic feature: for any logical connective there should be proof rules explaining how to introduce that connective on the left and right of the conclusion of the rule.
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**Characteristic feature:** for any logical connective there should be proof rules explaining how to introduce that connective on the left and right of the conclusion of the rule.

There are also structural rules that only involve sequent structure, not logical connectives.
Example: Gentzen’s LK

E.g., in Gentzen’s LK for classical propositional logic, the sequents are built from sets, interpreted as

\[ \Gamma \vdash \Delta \text{ is valid } \iff \bigwedge \Gamma \vdash \bigvee \Delta \]
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and the rules for $\to$ are:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \quad (\to L) \quad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \quad (\to R)$$
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and the rules for $\rightarrow$ are:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \quad (\rightarrow L) \quad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \quad (\rightarrow R)$$

Structural rules include:

$$\frac{\Gamma, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \quad (\text{ContrL}) \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Delta'} \quad (\text{WkL})$$
Analyticity

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This means getting rid of the dreaded cut rule, the sequent equivalent of modus ponens:

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\]

Getting rid of this is called cut-elimination, and proof theorists are absolutely obsessed with it!
Recall:

**Provability in BBI** is given by extending a Hilbert system for propositional classical logic by

\[
\begin{align*}
A \ast B & \vdash B \ast A \\
A \ast (B \ast C) & \vdash (A \ast B) \ast C \\
A & \vdash A \ast I \\
A \ast I & \vdash A \\
A_1 \vdash B_1 & \quad A_2 \vdash B_2 \\
\hline
A_1 \ast A_2 & \vdash B_1 \ast B_2 \\
A \ast B & \vdash C \\
A \vdash B \ast C \\
A \ast B & \vdash C
\end{align*}
\]
Motivation

• Can we give an analytic proof system for BBI?
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• For quite a long time in the 2000s, researchers tried to find a nice sequent calculus for BBI, but cut-elimination typically failed.

• But we can give an analytic Gentzen system based on the slightly more general notion of display calculus.
Display calculus: an overview

- **Display calculi** were first formulated by Belnap in the 1980s (sequent calculi were invented by Gentzen in the 1930s).
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- But, the structures $X$ and $Y$ can be structurally more complex than simple sets or multisets.
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- But, the structures $X$ and $Y$ can be structurally more complex than simple sets or multisets.

- Most importantly, display calculi allow us to rearrange sequents to focus on any individual part (like rearranging an equation in standard algebra).
Structures and interpretation

Structures $X$ defined as follows:

$$X ::= A \mid \emptyset \mid \#X \mid X; X \mid X, X$$
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A sequent $X \vdash Y$ is valid if $\Psi_X \models \Upsilon_Y$, (N.B. (1) we switch from one interpretation function to the other when going inside $\#$; (2) $\emptyset$ is not allowed to occur "positively" in a sequent.)
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A sequent $X \vdash Y$ is valid if $\Psi_X \models \Upsilon_Y$, where $\Psi_\_ \text{ and } \Upsilon_\_$ are defined by:

$$\Psi_A = A \quad \Upsilon_A = A$$
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\[
\begin{align*}
\Psi_A &= A \\
\Psi_{\emptyset} &= I \\
\Psi_{\#X} &= \neg \Upsilon_X \\
\Psi_{X; X} &= \Psi_X \land \Psi_Y \\
\Psi_{X, X} &= \Psi_X \lor \Psi_Y \\
\Psi_{X; Y} &= \Psi_X^* \Psi_Y \\
\Psi_{X, Y} &= \Psi_X^* \Upsilon_Y \\
\Upsilon_A &= A \\
\Upsilon_{\emptyset} &= \text{undefined}
\end{align*}
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\Psi_{X,Y} &= \Psi_X \ast \Psi_Y
\end{align*}$$

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Display property

We give the following display rules for our sequents:

\[ X ; Y \vdash Z \leftrightarrow_D X \vdash \#Y ; Z \leftrightarrow_D Y ; X \vdash Z \]
**Display property**

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\[
\begin{align*}
X ; Y &\vdash Z \iff_D X \vdash Y ; Z \iff_D Y ; X \vdash Z \\
X &\vdash Y ; Z \iff_D X ; ^\# Y &\vdash Z \iff_D X \vdash Z ; Y
\end{align*}
\]
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We give the following display rules for our sequents:

\[
\begin{align*}
    X ; Y \vdash Z & \iff_D X \vdash \#Y ; Z \iff_D Y ; X \vdash Z \\
    X \vdash Y ; Z & \iff_D X ; \#Y \vdash Z \iff_D X \vdash Z ; Y \\
    X \vdash Y & \iff_D \#Y \vdash \#X \iff_D \#\#X \vdash Y
\end{align*}
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\[
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X ; Y & \vdash Z \quad \iff_D X \vdash \#Y ; Z \quad \iff_D Y ; X \vdash Z \\
X & \vdash Y ; Z \quad \iff_D X ; \#Y \vdash Z \quad \iff_D X \vdash Z ; Y \\
X & \vdash Y \quad \iff_D \#Y \vdash \#X \quad \iff_D \#\#X \vdash Y \\
X , Y & \vdash Z \quad \iff_D X \vdash Y , Z \quad \iff_D Y , X \vdash Z
\end{align*}
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\[
\begin{align*}
X ; Y \vdash Z & \quad <>_D \quad X \vdash \#Y ; Z \quad <>_D \quad Y ; X \vdash Z \\
X \vdash Y ; Z & \quad <>_D \quad X ; \#Y \vdash Z \quad <>_D \quad X \vdash Z ; Y \\
X \vdash Y & \quad <>_D \quad \#Y \vdash \#X \quad <>_D \quad \#X \vdash Y \\
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We call the reflexive-transitive closure of these rules display equivalence, \(\equiv_D\).
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\[
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X; Y \vdash Z & \iff D \quad X \vdash \# Y; Z \iff D \quad Y; X \vdash Z \\
X \vdash Y; Z & \iff D \quad X; \# Y \vdash Z \iff D \quad X \vdash Z; Y \\
X \vdash Y & \iff D \quad \# Y \vdash \# X \iff D \quad \# \# X \vdash Y \\
X, Y \vdash Z & \iff D \quad X \vdash Y, Z \iff D \quad Y, X \vdash Z
\end{align*}
\]

We call the reflexive-transitive closure of these rules display equivalence, \( \equiv_D \). Then we get the crucial display property:

**Theorem**

For any “negative” part \( Z \) of \( X \vdash Y \) we have \( X \vdash Y \equiv_D Z \vdash W \), and for any “positive” part \( Z \) of \( X \vdash Y \) we have \( X \vdash Y \equiv_D W \vdash Z \).
Identity and logical rules

Identity rules:

\[
\begin{align*}
\frac{A}{A} & \quad \text{(Id)} \\
\frac{W \vdash Z}{X \vdash Y} & \quad W \vdash Z \equiv_D X \vdash Y (\equiv_D) \\
\frac{X \vdash Y}{X \vdash Y} & \quad \text{(Cut)}
\end{align*}
\]

Logical rules:

\[
\begin{align*}
\frac{A \vdash X \quad B \vdash X}{A \lor B \vdash X} & \quad (\lor L) \\
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X ; Y} & \quad (\rightarrow L) \\
\frac{X \vdash A \quad B \vdash Y}{A \neg* B \vdash X , Y} & \quad (\neg*L) \\
\frac{X \vdash A \quad B \vdash Y}{A \neg* B \vdash X , Y} & \quad (\neg*R)
\end{align*}
\]

(etc.)
Structural rules

\[
\frac{X ; X \vdash Z}{X \vdash Z} \quad (\text{Contr}) \quad \frac{X \vdash Z}{X ; Y \vdash Z} \quad (\text{Weak})
\]

\[
\frac{X \vdash Y}{\emptyset, X \vdash Y} \quad (\emptyset 1) \quad \frac{\emptyset, X \vdash Y}{X \vdash Y} \quad (\emptyset 2) \quad \frac{W, (X, Y) \vdash Z}{(W, X), Y \vdash Z} \quad (\text{Assoc})
\]
Soundness

Theorem (Soundness)

If $X \vdash Y$ is provable in our display calculus then it is valid.
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E.g., for the rule

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\frac{X \vdash A \quad B \vdash Y}{A \rightarrow^* B \vdash X, Y} \quad (\rightarrow^* \text{L})
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$$
\frac{X \vdash A \quad B \vdash Y}{A \to \top \vdash X, Y} \quad (\to\text{L})
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assume premises are valid, i.e. $\Psi_X \models A$ and $B \models \Upsilon_Y$; we have to show $A \to \top \models \Psi_X \to \Upsilon_Y$. 

**Soundness**

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E.g., for the rule

\[
\frac{X \vdash A \quad B \vdash Y}{A \rightarrow \dashv B \vdash X, Y} \tag{\neg \ast L}
\]

assume premises are valid, i.e. $\Psi_X \models A$ and $B \models \Upsilon_Y$; we have to show $A \rightarrow \dashv B \models \Psi_X \rightarrow \Upsilon_Y$.

This can be done by appealing to the semantics, or by deriving in the Hilbert system for BBI.
Completeness (1)

Theorem
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Lemma (1)
For any structure $X$, both $X \vdash \Psi_X$ and $\Upsilon_X \vdash X$ are provable.
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**Theorem**
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**Lemma (1)**
For any structure $X$, both $X \vdash \Psi_X$ and $\Upsilon_X \vdash X$ are provable.
(Proof by structural induction on $X$. Note we only care about the case where $\Upsilon_X$ is defined.)
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(Proof by structural induction on $X$. Note we only care about the case where $\Upsilon_X$ is defined.)

**Lemma (2)**
If $F \vdash G$ is provable in the Hilbert system for BBI then it is provable in the display calculus too.
Proof of completeness

Suppose $X \vdash Y$ is valid, i.e. $\Psi_X \models \Upsilon_Y$. 
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Suppose $X \vdash Y$ is valid, i.e. $\Psi_X \models \Upsilon_Y$.

By completeness of Hilbert system, $\Psi_X \vdash \Upsilon_Y$ is provable in BBI.
Proof of completeness

Suppose \( X \vdash Y \) is valid, i.e. \( \Psi_X \models \gamma_Y \).

By completeness of Hilbert system, \( \Psi_X \vdash \gamma_Y \) is provable in BBI.

Then \( X \vdash Y \) is provable in display calculus as follows:
Suppose \( X \vdash Y \) is valid, i.e. \( \Psi_X \models \Upsilon_Y \).

By completeness of Hilbert system, \( \Psi_X \vdash \Upsilon_Y \) is provable in BBI.

Then \( X \vdash Y \) is provable in display calculus as follows:

\[
\begin{array}{c}
\text{(Lemma 1)} \\
\vdots \\
X \vdash \Psi_X \\
\hline
\Psi_X \vdash \Upsilon_Y \\
\Upsilon_Y \vdash Y \\
\hline
\Psi_X \vdash Y \\
\hline
X \vdash Y
\end{array}
\]

L(H)
Cut-elimination

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**Theorem (Cut-elimination)**

Any proof of $X \vdash Y$ can be transformed into a proof of $X \vdash Y$ without (Cut):

$$
\begin{array}{c}
X \vdash F \\
F \vdash Y
\end{array}
\quad (\text{Cut})
$$

Belnap '82 famously gave a set of syntactic conditions C1–C8 on the proof rules of a display calculus which are sufficient to guarantee this. Most are boring and easy to check. The only non-trivial one is that so-called principal cuts can be reduced to cuts on smaller formulas.
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**Theorem (Cut-elimination)**

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\frac{X \vdash F \quad F \vdash Y}{X \vdash Y} \quad \text{(Cut)}
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**Cut-elimination**

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**Theorem (Cut-elimination)**

*Any proof of* $X \vdash Y$ *can be transformed into a proof of* $X \vdash Y$ *without (Cut):*

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X \vdash Y
\end{align*}
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*Belnap ’82 famously gave a set of syntactic conditions C1–C8 on the proof rules of a display calculus which are **sufficient to guarantee** this.*

*Most are boring and easy to check. The only non-trivial one is that so-called **principal cuts** can be reduced to cuts on smaller formulas.*
An instance of cut in a proof is called principal if the cut formula $F$ has immediately been introduced in both premises by the right- and left-side logical rules for the main connective in $F$. 
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E.g., the following is a principal cut:

\[
\begin{align*}
X \vdash F & , G \\
Y \vdash F & G \vdash Z \\
\hline
X \vdash F \not* G & \quad Y \vdash F \quad G \vdash Z \\
\hline
X \vdash Y & , Z \\
\end{align*}
\]

(Belnap's condition C8 requires us to show that we can transform this derivation into one where only cuts on the smaller subformulas, $F$ and $G$, are used.)
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$$
\frac{
X \vdash F, G \\
Y \vdash F \quad G \vdash Z
}{
X \vdash F \rightarrow G \quad F \rightarrow G \vdash Y \quad Z
} \quad (\rightarrow R) \quad (\rightarrow L)

\frac{
X \vdash Y \quad Z
}{
X \vdash Y, Z
} \quad (\text{Cut})
$$

Belnap’s condition C8 requires us to show that we can transform this derivation into one where only cuts on the smaller subformulas, $F$ and $G$, are used.
Here’s the reduced principal cut:

\[
\frac{X \vdash F, G}{X, F \vdash G} \quad (D\equiv) \\
\frac{X, F \vdash G \quad G \vdash Z}{X, F \vdash Z} \quad (\text{Cut})
\]

\[
\frac{Y \vdash F \quad F \vdash X, Z}{Y \vdash X, Z} \quad (D\equiv) \\
\frac{Y \vdash X, Z}{X \vdash Y, Z} \quad (D\equiv)
\]

Other types of principal cut can be treated similarly. This gives us cut-elimination by Belnap’s theorem.
Consequences

- Proof search in this system, even though it’s analytic, is still very difficult (display rules, structural rules).
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- In general, for both display and sequent calculi:
  
  cut-elimination $\nRightarrow$ (semi)decidability
  
  (cf. linear logic, relevant logic, arithmetic . . . )
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• Indeed, as we shall see in the next lecture, BBI is still in fact undecidable.
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- In general, for both display and sequent calculi:
  
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  (cf. linear logic, relevant logic, arithmetic ...)

- Indeed, as we shall see in the next lecture, BBI is still in fact undecidable.

- Cut-elimination provides structure and removes infinite branching points from the proof search space.
Further reading

James Brotherston.
Bunched logics displayed.

Nuel D. Belnap, Jr.
Display logic.

D. Larchey-Wendling and D. Galmiche.
Exploring the relation between intuitionistic BI and Boolean BI: an unexpected embedding.

J. Park, J. Seo and S. Park.
A theorem prover for Boolean BI.

Z. Hóu, A. Tiu and R. Goré.
A labelled sequent calculus for BBI: proof theory and proof search.