Boolean bunched logic: its semantics and completeness

James Brotherston

Programming Principles, Logic and Verification Group
Dept. of Computer Science
University College London, UK
J.Brotherston@ucl.ac.uk

Logic Summer School, ANU, 8 December 2015

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- Formulas can be understood as sets of “worlds” (often “resources”) in an underlying model.
- The multiplicatives generally denote composition operations on these worlds.
- Bunched logics are closely related to relevant logics and can also be seen as modal logics.

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- The multiplicatives can be seen as modalities in modal logic (more on that later).

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- $A * B$ can be read as "my current resource decomposes into two parts that satisfy A and B respectively".
- I can be read as "my resource is empty / of unit type".
- $A \mathrel{-\!\!*} B$ can be read as "if I add a resource satisfying A to my current resource, the whole thing satisfies B".

BBI, *proof-theoretically*

Provability in BBI is given by extending a Hilbert system for propositional classical logic by

$$A * B \vdash B * A \qquad A * (B * C) \vdash (A * B) * C$$

$$A \vdash A * \mathrm{I} \qquad A * \mathrm{I} \vdash A$$

$$\frac{A_1 \vdash B_1 \quad A_2 \vdash B_2}{A_1 * A_2 \vdash B_1 * B_2} \qquad \frac{A * B \vdash C}{A \vdash B \mathrel{-\!\!*} C} \qquad \frac{A \vdash B \mathrel{-\!\!*} C}{A * B \vdash C}$$

These rules are exactly the usual ones for multiplicative intuitionistic linear logic (MILL).

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(Note that $\circ$ can equivalently be seen as a ternary relation, $\circ \subseteq W \times W \times W$.)

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A is valid in M iff $w \models_\rho A$ for all ρ and $w \in W$.

Soundness and completeness

Theorem (Galmiche and Larchey-Wendling, 2006)

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- *Soundness* ($\Rightarrow$) is straightforward: just show that each proof rule preserves validity. (Easy exercise!)
- *Completeness* ($\Leftarrow$) is much harder.
- Several different approaches are possible; I am going to try to show you the simplest one, based on the Sahlqvist completeness theorem for modal logic.

Outline of the approach

- We translate BBI into a normal modal logic over "diamond" modalities I, $*$, $\multimap$, satisfying a set of well-behaved Sahlqvist axioms. $A \multimap B$ will come out as $\neg(A \mathrel{-\!\!*} \neg B)$.

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A **ML**$_{\text{BBI}}$ frame is given by $\langle W, \circ, \multimap, E\rangle$, where $E \subseteq W$ and $\circ, \multimap \colon W \times W \to \mathcal{P}(W)$ (like in BBI).

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A is *valid* in M iff $w \models_\rho A$ for all $w \in W$ and valuations ρ.
Same as BBI, except for $-\!\!*$ versus $\multimap$!

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(3) & A * B \vdash B * A \\
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Theorem (Sahlqvist)

If B is valid in the $\mathbf{ML}_{\mathrm{BBI}}$ frames satisfying $\mathcal{A}_{\mathrm{BBI}}$, then it is provable in $\mathbf{ML}_{\mathrm{BBI}} + \mathcal{A}_{\mathrm{BBI}}$.

Modal frames are BBI*-models*

Lemma (1)

Let $M = \langle W, \circ, -\!\!\circ, E \rangle$ *be a modal frame satisfying axioms (1) and (2) of* $\mathcal{A}_{\mathrm{BBI}}$. *Then we have, for any* $w, w_1, w_2 \in W$*:*

$$w \in w_1 \multimap w_2 \quad \Leftrightarrow \quad w_2 \in w \circ w_1.$$

Proof.

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Hint: $(\Leftarrow)$ uses axiom (1), $(\Rightarrow)$ uses axiom 2.

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Proof.

Hint: $(\Leftarrow)$ uses axiom (1), $(\Rightarrow)$ uses axiom 2. □

So, when axioms (1) and (2) are satisfied, Lemma 1 gives us:

$$w \models_\rho A \multimap B \quad \Leftrightarrow \quad w \in w_1 \multimap w_2 \text{ and } w_1 \models_\rho A \text{ and } w_2 \models_\rho B$$

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$$w \in w_1 \multimap w_2 \quad \Leftrightarrow \quad w_2 \in w \circ w_1.$$

Proof.

Hint: $(\Leftarrow)$ uses axiom (1), $(\Rightarrow)$ uses axiom 2. □

So, when axioms (1) and (2) are satisfied, Lemma 1 gives us:

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Modal frames are BBI-*models*

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Translating between BBI *and* $\mathbf{ML}_{\mathrm{BBI}}$

- Given a BBI-formula A, write $t(A)$ for the $\mathbf{ML}_{\mathrm{BBI}}$ formula obtained by replacing every formula of the form $B \twoheadrightarrow C$ by $\neg(B \multimap \neg C)$.

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Proof.

Structural induction on A. □

Validity translation lemma

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Let $M = \langle W, \circ, \multimap, E \rangle$ *be a* $\mathbf{ML}_{\mathrm{BBI}}$ *frame satisfying axioms (3)–(6) of* $\mathcal{A}_{\mathrm{BBI}}$*. Then* $\langle W, \circ, E \rangle$ *is a* BBI-*model.*

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Lemma (4)

If A *is valid in* BBI, *then* $t(A)$ *is valid in every* $\mathbf{ML}_{\mathrm{BBI}}$ *frame satisfying* $\mathcal{A}_{\mathrm{BBI}}$.

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Uses Lemmas 1 and 3. □

Proof translation lemma

Lemma (5)

If B is provable in $\mathbf{ML}_{\mathrm{BBI}} + \mathcal{A}_{\mathrm{BBI}}$, then $u(B)$ is provable in BBI.

Proof translation lemma

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If B is provable in $\mathbf{ML}_{\mathrm{BBI}} + \mathcal{A}_{\mathrm{BBI}}$, then $u(B)$ is provable in BBI*.*

Proof.

By induction on the proof of B in $\mathbf{ML}_{\mathrm{BBI}} + \mathcal{A}_{\mathrm{BBI}}$. We have to show that every proof rule in $\mathbf{ML}_{\mathrm{BBI}} + \mathcal{A}_{\mathrm{BBI}}$ is derivable in BBI under the translation $u(-)$. □

Proof of completeness

Theorem

If A is BBI-*valid then it is* BBI-*provable.*

Proof.

Let A be BBI-valid.

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Finally, by Lemma 2, A is provable in BBI. □

Exercise: fill in the proofs of Lemmas 1–5!

Further reading

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