Boolean bunched logic: its semantics and completeness

James Brotherston

Programming Principles, Logic and Verification Group
Dept. of Computer Science
University College London, UK
J.Brotherston@ucl.ac.uk

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Bunched logics

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- **Bunched logics** extend classical or intuitionistic logic with various “linear” or **multiplicative** connectives.

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- Bunched logics are closely related to relevant logics and can also be seen as modal logics.
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- *, a multiplicative **conjunction**;

*“Multiplicative” means* $\cdot$ *does not satisfy weakening or contraction:*

$$A \cdot B \not\vdash A$$

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The multiplicatives can be seen as modalities in modal logic (more on that later).
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  • I, a multiplicative unit.

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The multiplicatives can be seen as **modalities** in modal logic (more on that later).
Intuitively, formulas in BBI can be read as properties of resources.

• $A \cdot B$ can be read as "my current resource decomposes into two parts that satisfy $A$ and $B$ respectively".

• $I$ can be read as "my resource is empty / of unit type".

• $A \vdash I$ can be read as "if I add a resource satisfying $A$ to my current resource, the whole thing satisfies $B$".
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- $A \rightarrow B$ can be read as “if I add a resource satisfying $A$ to my current resource, the whole thing satisfies $B$”.
BBI, *proof-theoretically*

**Provability** in BBI is given by extending a Hilbert system for propositional classical logic by

\[
A \ast B \vdash B \ast A \quad A \ast (B \ast C) \vdash (A \ast B) \ast C
\]

\[
A \vdash A \ast I \quad A \ast I \vdash A
\]

\[
A_1 \vdash B_1 \quad A_2 \vdash B_2 \quad A \ast B \vdash C \quad A \vdash B \supset C
\]

\[
A_1 \ast A_2 \vdash B_1 \ast B_2 \quad A \vdash B \supset C \quad A \ast B \vdash C
\]

These rules are exactly the usual ones for *multiplicative intuitionistic linear logic* (MILL).
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$$W_1 \circ W_2 \overset{\text{def}}{=} \bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 \circ w_2$$

(Note that $\circ$ can equivalently be seen as a ternary relation, $\circ \subseteq W \times W \times W$.)

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- $\circ$ is **commutative** and **associative**;
- the set of **units** $E \subseteq W$ satisfies $w \circ E = \{w\}$ for all $w \in W$. 

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Given $M$, $\rho$, and $w \in W$, we define the forcing relation $w \models_{\rho} A$ by induction on formula $A$:

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A valuation for BBI-model \( M = \langle W, \circ, E \rangle \) is a function \( \rho \) from propositional variables to \( \mathcal{P}(W) \).
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w \models_\rho I & \iff w \in E
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- $w \models_{\rho} P \iff w \in \rho(P)$
- $w \models_{\rho} A \rightarrow B \iff w \models_{\rho} A$ implies $w \models_{\rho} B$
- $w \models_{\rho} \text{I} \iff w \in E$
- $w \models_{\rho} A \ast B \iff w \in w_1 \circ w_2$ and $w_1 \models_{\rho} A$ and $w_2 \models_{\rho} B$
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    w \models_\rho A \ast B & \iff \forall w', w'' \in W. \text{ if } w'' \in w \circ w' \text{ and } w' \models_\rho A \text{ then } w'' \models_\rho B
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    w \models_{\rho} A \ast B & \iff \forall w', w'' \in W. \text{ if } w'' \in w \circ w' \text{ and } w' \models_{\rho} A \\
    & \text{ then } w'' \models_{\rho} B
\end{align*}
\]

$A$ is valid in $M$ iff $w \models_{\rho} A$ for all $\rho$ and $w \in W$. 

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Soundness and completeness

Theorem (Galmiche and Larchey-Wendling, 2006)
A formula is BBI-provable iff it is valid in all BBI-models.

Soundness (⇒) is straightforward: just show that each proof rule preserves validity. (Easy exercise!)

Completeness (⇐) is much harder.

Several different approaches are possible; I am going to try to show you the simplest one, based on the Sahlqvist completeness theorem for modal logic.
Soundness and completeness

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• Several different approaches are possible; I am going to try to show you the simplest one, based on the Sahlqvist completeness theorem for modal logic.
Outline of the approach

- We translate BBI into a normal modal logic over “diamond” modalities $I, *, \circ, \neg\circ$, satisfying a set of well-behaved Sahlqvist axioms. $A \circ B$ will come out as $\neg(A \circ \neg B)$. 
Outline of the approach

• We translate BBI into a normal modal logic over “diamond” modalities I, ∗, →, satisfying a set of well-behaved Sahlqvist axioms. A → B will come out as ¬(A → ¬B).

• Then the Sahlqvist completeness theorem says that this modal logic is complete for its models in modal logic.
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• By suitable translations \( t \) and \( u \) between BBI and this modal logic, we get
  \( A \) valid in BBI
Outline of the approach

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- By suitable translations $t$ and $u$ between BBI and this modal logic, we get
  
  $A$ valid in BBI $\implies t(A)$ valid in modal logic
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• We translate BBI into a normal modal logic over “diamond” modalities I, *, →, satisfying a set of well-behaved Sahlqvist axioms. $A \rightarrow B$ will come out as $\neg (A \multimap \neg B)$.

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  $A$ valid in BBI $\Rightarrow$ $t(A)$ valid in modal logic
  $\Rightarrow$ $t(A)$ provable in modal logic (Sahlqvist)
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  \begin{align*}
  A \text{ valid in BBI} & \Rightarrow t(A) \text{ valid in modal logic} \\
  & \Rightarrow t(A) \text{ provable in modal logic (Sahlqvist)} \\
  & \Rightarrow u(t(A)) \text{ provable in BBI} \\
  & \Rightarrow A \text{ provable in BBI}
  \end{align*}
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BBI as a modal logic

A $\textbf{ML}_{\text{BBI}}$ formula is built from propositional variables using the classical connectives, constant I and binary modalities $\ast$ and $\to$. 
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Provability in the \textit{normal modal logic} for \textbf{ML}_{\text{BBI}} is given by extending classical propositional logic with the following axioms and rules, where $\otimes \in \{\ast, \rightarrow\}$:
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$$\bot \otimes A \vdash \bot \quad \text{and} \quad A \otimes \bot \vdash \bot$$

$$(A \lor B) \otimes C \vdash (A \otimes C) \lor (B \otimes C)$$

$$A \otimes (B \lor C) \vdash (A \otimes B) \lor (A \otimes C)$$

$$\frac{A_1 \vdash A_2 \quad B_1 \vdash B_2}{A_1 \otimes B_1 \vdash A_2 \otimes B_2}$$
A $\textbf{ML}_{\text{BBI}}$ frame is given by $\langle W, \circ, \triangleright, E \rangle$, where $E \subseteq W$ and $\circ, \triangleright: W \times W \rightarrow \mathcal{P}(W)$ (like in BBI).
BBI as a modal logic (2)

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Now we give the forcing relation \( w \models_{\rho} A \):

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w \models_{\rho} P \iff w \in \rho(P)
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   w \models_{\rho} A \rightarrowo B & \iff w \in w_1 \rightarrowo w_2 \text{ and } w_1 \models_{\rho} A \text{ and } w_2 \models_{\rho} B
\end{align*}
\]

\( A \) is valid in \( M \) iff \( w \models_{\rho} A \) for all \( w \in W \) and valuations \( \rho \).

Same as BBI, except for \( \ast \) versus \( \rightarrowo \)!
Sahlqvist axioms for $\text{ML}_{\text{BBI}}$

Define a set $\mathcal{A}_{\text{BBI}}$ of $\text{ML}_{\text{BBI}}$-formulas as follows:
Define a set $A_{BBI}$ of $ML_{BBI}$-formulas as follows:

1. $A \land (B * C) \vdash (B \land (C \rightarrow A)) * \top$
2. $A \land (B \rightarrow C) \vdash \top \rightarrow (C \land (A * B))$
3. $A * B \vdash B * A$
4. $A * (B * C) \vdash (A * B) * C$
5. $A * I \vdash A$
6. $A \vdash A * I$

These are all of a form called Sahlqvist formulas, and so we have by the Sahlqvist completeness theorem:

Theorem (Sahlqvist)
If $B$ is valid in the $ML_{BBI}$ frames satisfying $A_{BBI}$, then it is provable in $ML_{BBI}$.
Define a set $\mathcal{A}_{\text{BBI}}$ of $\text{ML}_{\text{BBI}}$-formulas as follows:

\begin{align*}
(1) & \quad A \land (B \star C) \vdash (B \land (C \rightarrow A)) \star \top \\
(2) & \quad A \land (B \rightarrow C) \vdash \top \rightarrow (C \land (A \star B)) \\
(3) & \quad A \star B \vdash B \star A \\
(4) & \quad A \star (B \star C) \vdash (A \star B) \star C \\
(5) & \quad A \star I \vdash A \\
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\end{align*}

These are all of a form called Sahlqvist formulas, and so we have by the Sahlqvist completeness theorem:
Sahlqvist axioms for $\mathbf{ML}_{\text{BBI}}$

Define a set $\mathcal{A}_{\text{BBI}}$ of $\mathbf{ML}_{\text{BBI}}$-formulas as follows:

1. $A \land (B \ast C) \vdash (B \land (C \to A)) \ast \top$
2. $A \land (B \to C) \vdash \top \to (C \land (A \ast B))$
3. $A \ast B \vdash B \ast A$
4. $A \ast (B \ast C) \vdash (A \ast B) \ast C$
5. $A \ast I \vdash A$
6. $A \vdash A \ast I$

These are all of a form called Sahlqvist formulas, and so we have by the Sahlqvist completeness theorem:

**Theorem (Sahlqvist)**

*If $B$ is valid in the $\mathbf{ML}_{\text{BBI}}$ frames satisfying $\mathcal{A}_{\text{BBI}}$, then it is provable in $\mathbf{ML}_{\text{BBI}} + \mathcal{A}_{\text{BBI}}$.***
Modal frames are BBI-models

Lemma (1)
Let $M = \langle W, \circ, \rightarrow, E \rangle$ be a modal frame satisfying axioms (1) and (2) of $\mathcal{A}_{\text{BBI}}$. Then we have, for any $w, w_1, w_2 \in W$:

$$w \in w_1 \rightarrow w_2 \iff w_2 \in w \circ w_1.$$ 

Proof.
Modal frames are BBI-models

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Proof.

Hint: $(\Leftarrow)$ uses axiom (1), $(\Rightarrow)$ uses axiom 2.
**Modal frames are BBI-models**

**Lemma (1)**

Let $M = \langle W, \circ, \rightarrow, E \rangle$ be a modal frame satisfying axioms (1) and (2) of $A_{BBI}$. Then we have, for any $w, w_1, w_2 \in W$:

$$w \in w_1 \rightarrow w_2 \iff w_2 \in w \circ w_1.$$ 

**Proof.**

Hint: $(\Leftarrow)$ uses axiom (1), $(\Rightarrow)$ uses axiom 2.

So, when axioms (1) and (2) are satisfied, Lemma 1 gives us:

$$w \models_{\rho} A \rightarrow B \iff w \in w_1 \rightarrow w_2 \text{ and } w_1 \models_{\rho} A \text{ and } w_2 \models_{\rho} B.$$
**Modal frames are BBI-models**

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\[
\iff w \models_\rho \neg(A \rightarrow \neg B)
\]
Translating between BBI and $\text{ML}_{\text{BBI}}$

- Given a BBI-formula $A$, write $t(A)$ for the $\text{ML}_{\text{BBI}}$ formula obtained by replacing every formula of the form $B \rightarrow C$ by $\lnot (B \rightarrow \lnot C)$.

Lemma (2)
If $u(t(A))$ is provable in BBI then so is $A$.

Proof. Structural induction on $A$. 

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Translating between BBI and $\text{ML}_{\text{BBI}}$

- Given a BBI-formula $A$, write $t(A)$ for the $\text{ML}_{\text{BBI}}$ formula obtained by replacing every formula of the form $B \multimap C$ by $\neg (B \multimap \neg C)$.

- Conversely, given $\text{ML}_{\text{BBI}}$ formula $A$, write $u(A)$ for the BBI-formula obtained by replacing every formula of the form $B \multimap C$ by $\neg (B \multimap \neg C)$.

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Translating between BBI and $\text{ML}_{\text{BBI}}$

- Given a BBI-formula $A$, write $t(A)$ for the $\text{ML}_{\text{BBI}}$ formula obtained by replacing every formula of the form $B \imp C$ by $\neg(B \imp \neg C)$.
- Conversely, given $\text{ML}_{\text{BBI}}$ formula $A$, write $u(A)$ for the BBI-formula obtained by replacing every formula of the form $B \imp C$ by $\neg(B \imp \neg C)$.

Lemma (2)

If $u(t(A))$ is provable in BBI then so is $A$.

Proof.

Structural induction on $A$. 

\[\square\]
Validity translation lemma

**Lemma (3)**

Let $M = \langle W, \circ, \neg, E \rangle$ be a $\text{ML}_{\text{BBI}}$ frame satisfying axioms (3)–(6) of $\mathcal{A}_{\text{BBI}}$. Then $\langle W, \circ, E \rangle$ is a BBI-model.
Lemma (3)

Let $M = \langle W, \odot, \rightarrow, E \rangle$ be a $\text{ML}_{\text{BBI}}$ frame satisfying axioms (3)–(6) of $A_{\text{BBI}}$. Then $\langle W, \odot, E \rangle$ is a BBI-model.

Proof.

Easy exercise!
Validity translation lemma

Lemma (3)

Let \( M = \langle W, \circ, \rightarrow, E \rangle \) be a \( \text{ML}_{\text{BBI}} \) frame satisfying axioms (3)–(6) of \( \mathcal{A}_{\text{BBI}} \). Then \( \langle W, \circ, E \rangle \) is a BBI-model.

Proof.

Easy exercise!

Lemma (4)

If \( A \) is valid in BBI, then \( t(A) \) is valid in every \( \text{ML}_{\text{BBI}} \) frame satisfying \( \mathcal{A}_{\text{BBI}} \).
Validity translation lemma

Lemma (3)
Let $M = \langle W, \circ, \rightarrow, E \rangle$ be a $\text{ML}_{\text{BBI}}$ frame satisfying axioms (3)–(6) of $A_{\text{BBI}}$. Then $\langle W, \circ, E \rangle$ is a BBI-model.

Proof.
Easy exercise!

Lemma (4)
If $A$ is valid in BBI, then $t(A)$ is valid in every $\text{ML}_{\text{BBI}}$ frame satisfying $A_{\text{BBI}}$.

Proof.
Uses Lemmas 1 and 3.
Proof translation lemma

Lemma (5)

If $B$ is provable in $\text{ML}_{\text{BBI}} + A_{\text{BBI}}$, then $u(B)$ is provable in $\text{BBI}$. 
Lemma (5)

If $B$ is provable in $\text{ML}_{\text{BBI}} + \mathcal{A}_{\text{BBI}}$, then $u(B)$ is provable in $\text{BBI}$.

Proof.

By induction on the proof of $B$ in $\text{ML}_{\text{BBI}} + \mathcal{A}_{\text{BBI}}$. We have to show that every proof rule in $\text{ML}_{\text{BBI}} + \mathcal{A}_{\text{BBI}}$ is derivable in $\text{BBI}$ under the translation $u(\cdot)$. 

\qed
Proof of completeness

Theorem
If $A$ is BBI-valid then it is BBI-provable.

Proof.
Let $A$ be BBI-valid.
Proof of completeness

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By Lemma 4, $t(A)$ is valid in the class of $\text{ML}_{\text{BBI}}$ frames satisfying $A_{\text{BBI}}$. 

Proof of completeness

**Theorem**
If $A$ is BBI-valid then it is BBI-provable.

**Proof.**
Let $A$ be BBI-valid.

By Lemma 4, $t(A)$ is valid in the class of $\text{ML}_{\text{BBI}}$ frames satisfying $\mathcal{A}_{\text{BBI}}$.

By the Sahlqvist Theorem, $t(A)$ is provable in $\text{ML}_{\text{BBI}} + \mathcal{A}_{\text{BBI}}$. 
Proof of completeness

Theorem
If \( A \) is BBI-valid then it is BBI-provable.

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By Lemma 5, \( u(t(A)) \) is provable in BBI.
Proof of completeness

Theorem
If $A$ is BBI-valid then it is BBI-provable.

Proof.
Let $A$ be BBI-valid.

By Lemma 4, $t(A)$ is valid in the class of $\text{ML}_{\text{BBI}}$ frames satisfying $A_{\text{BBI}}$.

By the Sahlqvist Theorem, $t(A)$ is provable in $\text{ML}_{\text{BBI}} + A_{\text{BBI}}$.

By Lemma 5, $u(t(A))$ is provable in BBI.

Finally, by Lemma 2, $A$ is provable in BBI.
Proof of completeness

**Theorem**

*If* $A$ *is BBI-valid then it is BBI-provable.*

**Proof.**

Let $A$ be BBI-valid.

By Lemma 4, $t(A)$ is valid in the class of $\mathbf{ML}_{BBI}$ frames satisfying $A_{BBI}$.

By the Sahlqvist Theorem, $t(A)$ is provable in $\mathbf{ML}_{BBI} + A_{BBI}$.

By Lemma 5, $u(t(A))$ is provable in BBI.

Finally, by Lemma 2, $A$ is provable in BBI.

**Exercise:** fill in the proofs of Lemmas 1–5!
Further reading

D. Galmiche and D. Larchey-Wendling.
Expressivity properties of Boolean BI through relational models.

D. Pym.
The semantics and proof theory of the logic of bunched implications.

Context logic as modal logic: completeness and parametric inexpressivity.

J. Brotherston and J. Villard.
Sub-classical Boolean bunched logics and the meaning of par.