On the Complexity of Pointer Arithmetic in Separation Logic

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ADSL workshop (FLoC), University of Oxford, July 2018

Overview

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  - separating implication ($-\!\!*$);
  - arrays;
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- Our focus is on pointer arithmetic in SL.

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Question: *How much pointer arithmetic can one add to separation logic and remain within polynomial time?*

Minimal fragment, $\mathsf{SL}_{\mathsf{MPA}}$

- Terms t, pure formulas Π and spatial formulas F given by:

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\begin{array}{rcl}
t & ::= & x \mid x + k \mid \mathsf{nil} \\
\Pi & ::= & x = t \mid x \leq t \mid \Pi \wedge \Pi \\
F & ::= & \mathsf{emp} \mid t \mapsto t \mid F * F
\end{array}
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where $x \in \mathsf{Var}$, $k \in \mathbb{Z}$.

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- Symbolic heaps given by $\exists \mathbf{x}.\ \Pi : F$.
- Semantics given as usual by $s, h \models A$ in a stack-and-heap model over locations $\mathbb{N}$ and values $\mathbb{N} \cup \{nil\}$.

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$$\left.\begin{array}{l} x_1 \leq x_2 + k_1, \\ \ldots \\ x_{m-1} \leq x_m + k_{m-1}, \\ x_m \leq x_1 + k_m \end{array}\right\} \Rightarrow \ x_1 - x_1 \leq \sum_{i=1}^{m} k_i$$

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(The latter fails if we allow pointer sums, $x \leq y + z$.)

Some known upper bounds

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- Quantified entailment is in Π_1^{EXP}.

Satisfiability, lower bound

In fact, the lower bound for satisfiability is also NP.

3-colourability problem (NP-hard)

Given an undirected graph, decide whether there is a "perfect" 3-colouring of the vertices, such that no two adjacent vertices share the same colour.

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First, choose numbers e_{ij} for each edge (v_i, v_j) such that $|e_{i'j'} - e_{ij}| \geq 4$ for any two distinct edges.

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Then encode in $\mathsf{SL}_{\mathsf{MPA}}$ as (slightly simplified)

$$\bigwedge_{i=1}^{n} 1 \leq c_i \leq 3 : \; \mathop{\ast}_{(v_i,v_j)\in E} (c_i + e_{ij} \mapsto \mathsf{nil} * c_j + e_{ij} \mapsto \mathsf{nil})$$

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Note that $x \leq y + k \equiv \bot$ means $y \leq x - k - 1$.

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We will construct a model s, h of A in which all values are bounded by

$$M = \Sigma_{i=1}^{m} |k_i| + 1$$

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So s' satisfies our difference equation system, and thus γ_A. Then we can create a suitable h' with $s', h' \models A$.

Quantifier-free entailment

By relatively minor adaptations of the corresponding tricks for satisfiability, we have the following for the quantifier-free entailment problem:

1. a lower bound of coNP;
2. the small model property (any invalid entailment has a small countermodel).

Quantified entailment, lower bound

Lower bound is Π_2^P in the polynomial-time hierarchy.

2-round 3-colourability problem (Π_2^P-hard)

Given an undirected graph, decide whether every 3-colouring of the leaves can be extended to a perfect 3-colouring of the graph.

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Simplified encoding:

LHS: $\bigwedge_{i=1}^{k}(1 \leq c_i \leq 3)\colon \mathop{\ast}_{(v_i,v_j)\in E}^{\ell\in\{1,2,3\}} (e_{ij} + \ell) \mapsto \mathsf{nil}$

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RHS: $\exists \mathbf{z}.\ \bigwedge_{(v_i,v_j)\in E}^{1\leq k\leq n} (1 \leq c_k, \widetilde{c_{ij}} \leq 3)$:
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The key difference between Π^0_2 PbA and Π^P_2 is that in the latter all variables must be polynomially bounded.

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The key difference between Π_2^0 PbA and Π_2^P is that in the latter all variables must be polynomially bounded.

This follows from the small model property for quantified entailment. Construction uses similar ideas to satisfiability case, but is (quite a bit) more complex.

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- Quantified entailment is Π_2^P-complete.
- The small model property holds.

Thanks for listening!

Revised paper available (my webpage)