Craig Interpolation in Displayable Logics

James Brotherston[1] and Rajeev Goré[2]

[1] Dept. of Computing, Imperial College London
J.Brotherston@imperial.ac.uk
[2] School of Computer Science, ANU Canberra
Rajeev.Gore@anu.edu.au

Abstract. We give a general proof-theoretic method for proving Craig interpolation for displayable logics, based on an analysis of the individual proof rules of their display calculi. Using this uniform method, we prove interpolation for a spectrum of display calculi differing in their structural rules, including those for multiplicative linear logic, multiplicative additive linear logic and ordinary classical logic. Our analysis of proof rules also provides new insights into why interpolation fails, or seems likely to fail, in many substructural logics. Specifically, contraction appears particularly problematic for interpolation except in special circumstances.

1 Introduction

> I believe or hope that Display logic can be used as a basis for establishing an interpolation theorem; but that remains to be seen.
>
> Nuel D. Belnap, *Display Logic* [1], 1982

Craig's original *interpolation* theorem for first-order logic [6] states that for any provable entailment $F \vdash G$ between formulas, an "intermediate formula" or *interpolant* I can be found such that both $F \vdash I$ and $I \vdash G$ are provable and every nonlogical symbol occurring in I occurs in both F and G. This seemingly innocuous property turns out to have considerable mathematical significance because Craig interpolation is intimately connected with consistency, compactness and definability (see [8] for a survey). In computer science, it plays an important rôle in settings where modular decomposition of complex theories is a concern, and has been applied to such problems as invariant generation [16], type inference [12], model checking [5,15] and the decomposition of complex ontologies [13]. Whether a given logic satisfies interpolation is thus of practical importance in computer science as well as theoretical importance in logic.

In this paper, we give a proof-theoretic method for establishing Craig interpolation in the setting of Belnap's *display logic*. Display logic is a general consecution framework which allows us to combine multiple families of logical connectives into a single *display calculus* [1]. Display calculi are characterised by the availability of a "display-equivalence" relation on consecutions which allows us to rearrange a consecution so that a selected substructure appears alone on one side of the proof turnstile. Various authors have shown how to capture large

K. Brünnler and G. Metcalfe (Eds.): TABLEAUX 2011, LNAI 6793, pp. 88–103, 2011.
© Springer-Verlag Berlin Heidelberg 2011

classes of modal and substructural logics within this framework [2,11,14,20], and how to characterise the class of Kripke frame conditions that can be captured by displayed logics [10]. A major advantage of display calculi is that they enjoy an extremely general cut-elimination theorem which relies on checking eight simple conditions on the rules of the calculus. Restall has also shown how decidability results can be obtained from cut-free display calculi [17].

In the case that a cut-free sequent calculus *à la* Gentzen is available, interpolation for the logic in question can typically be established by induction over cut-free derivations (see e.g. [4]). Besides its theoretical elegance, this method has the advantage of being fully constructive. One of the main criticisms levelled against display calculi is that they do not enjoy a true sub-formula property and hence, in contrast to the situation for sequent calculi, Belnap's general cut-elimination theorem cannot be used to prove results like interpolation for display calculi. Indeed, to our knowledge there are no interpolation theorems for display calculi in the literature. Here we (partially) rebut the aforementioned criticism by giving a general Craig interpolation result for a large class of displayed logics.

The main idea of our approach is to construct a *set* of interpolants at each step of a given proof, one for every possible "rearrangement" of the consecution using both display-equivalence and any native associativity principles. Our aim is then to show that, given interpolants for all rearrangements of the premises of a rule, one can find interpolants for all rearrangements of its conclusion. This very general interpolation method applies to a wide range of logics with a display calculus presentation and is potentially extensible to even larger classes of such logics. However, some proof rules enjoy the aforementioned property only under strong restrictions, with contraction being the most problematic among the rules we study in this paper. This gives a significant new insight into the reasons why interpolation fails, or appears likely to fail, in many substructural logics.

Section 2 introduces the display calculi that we work with throughout the paper. We develop our interpolation methodology incrementally in Sections 3, 4 and 5. Section 6 concludes. The proofs in this paper have been abbreviated for space reasons; detailed proofs can be found in an associated technical report [3].

2 Display Calculus Fundamentals

We now give a basic display calculus which can be customised to various logics by adding structural rules. In general, one may formulate display calculi for logics involving arbitrarily many families of formula and structure connectives. To limit the bureaucracy and technical overhead due to such generality, we limit ourselves in this paper to display calculi employing only a single family of connectives. For similar reasons, we also restrict to commutative logics.

Definition 2.1 (Formula, Structure, Consecution). *Formulas* and *structures* are given by the following grammars, where P ranges over a fixed infinite set of propositional variables, F ranges over formulas, and X over structures:

$$F ::= P \mid \top \mid \bot \mid \neg F \mid F \,\&\, F \mid F \vee F \mid F \rightarrow F \mid \top_a \mid \bot_a \mid F \,\&_a\, F \mid F \vee_a F$$
$$X ::= F \mid \emptyset \mid \sharp X \mid X \,;\, X$$

(The subscript "a" is for "additive".) A structure is called *atomic* if it is either a formula or $\emptyset$. When we reason by induction on a structure X, we typically conflate the cases $X = F$ and $X = \emptyset$ into the case where X is atomic. We use F, G, I etc. to range over formulas, W, X, Y, Z etc. to range over structures, and A, B etc. to range over atomic structures. We write $\mathcal{V}(X)$ for the set of propositional variables occurring in the structure X. If X and Y are structures then $X \vdash Y$ is a *consecution*. We use $\mathcal{C}, \mathcal{C}'$ etc. to range over consecutions.

Definition 2.2 (Interpretation of structures). For any structure X we define the formulas Ψ_X and Υ_X by mutual structural induction on X as:

$$\begin{array}{llll} \Psi_F = F & \Psi_\emptyset = \top & \Upsilon_F = F & \Upsilon_\emptyset = \bot \\ \Psi_{\sharp X} = \neg \Upsilon_X & \Psi_{X_1;X_2} = \Psi_{X_1} \,\&\, \Psi_{X_2} & \Upsilon_{\sharp X} = \neg \Psi_X & \Upsilon_{X_1;X_2} = \Upsilon_{X_1} \vee \Upsilon_{X_2} \end{array}$$

For any consecution $X \vdash Y$ we define its *formula interpretation* to be $\Psi_X \vdash \Upsilon_Y$.

Definition 2.3 (Antecedent and consequent parts). A *part* of a structure X is an occurrence of one of its substructures. We classify the parts of X as either *positive* or *negative* in X as follows:

- X is a positive part of itself;
- a negative / positive part of X is a positive / negative part of $\sharp X$;
- a positive / negative part of X_1 or X_2 is a positive / negative part of X_1 ; X_2.

Z is said to be an *antecedent / consequent part* of a consecution $X \vdash Y$ if it is a positive / negative part of X or a negative / positive part of Y.

Definition 2.4 (Display-equivalence). We define *display-equivalence* $\equiv_D$ to be the least equivalence on consecutions containing the (symmetric) relation $\rightleftarrows_D$ given by the following *display postulates*:

$$\begin{array}{ccccc} X;Y \vdash Z & \rightleftarrows_D & X \vdash \sharp Y; Z & \rightleftarrows_D & Y;X \vdash Z \\ X \vdash Y;Z & \rightleftarrows_D & X;\sharp Y \vdash Z & \rightleftarrows_D & X \vdash Z;Y \\ X \vdash Y & \rightleftarrows_D & \sharp Y \vdash \sharp X & \rightleftarrows_D & \sharp\sharp X \vdash Y \end{array}$$

Note that Defn. 2.4 builds in the commutativity of ; on the left and right of consecutions, i.e., we are assuming both & and $\vee$ commutative.

Proposition 2.5 (Display property). *For any antecedent / consequent part Z of a consecution $X \vdash Y$, one can construct a structure W such that $X \vdash Y \equiv_D Z \vdash W$ / $X \vdash Y \equiv_D W \vdash Z$, respectively.*

Proof. (Sketch) For any $X \vdash Y$, the display postulates of Defn. 2.4 allow us to display each of the immediate substructures of X and Y (as the antecedent or consequent as appropriate). The proposition follows by iterating. □

Rearranging $X \vdash Y$ into $Z \vdash W$ or $W \vdash Z$ in Prop. 2.5 is called *displaying* Z.

Figure 1 gives the proof rules of a basic display calculus $\mathcal{D}_0$ which only uses the logical connectives $\top$, $\bot$, $\neg$, &, $\vee$, and $\rightarrow$. Figure 2 presents "structure-free" rules

Identity rules:

$$\frac{}{P \vdash P}\,(\text{Id}) \qquad \frac{X' \vdash Y'}{X \vdash Y}\ X \vdash Y \equiv_D X' \vdash Y'\ (\equiv_D)$$

Logical rules:

$$\frac{\emptyset \vdash X}{\top \vdash X}\,(\top\text{L}) \qquad \frac{}{\emptyset \vdash \top}\,(\top\text{R}) \qquad \frac{F; G \vdash X}{F \,\&\, G \vdash X}\,(\&\text{L}) \qquad \frac{X \vdash F \quad Y \vdash G}{X \,;\, Y \vdash F \,\&\, G}\,(\&\text{R})$$

$$\frac{}{\bot \vdash \emptyset}\,(\bot\text{L}) \qquad \frac{X \vdash \emptyset}{X \vdash \bot}\,(\bot\text{R}) \qquad \frac{F \vdash X \quad G \vdash Y}{F \vee G \vdash X \,;\, Y}\,(\vee\text{L}) \qquad \frac{X \vdash F; G}{X \vdash F \vee G}\,(\vee\text{R})$$

$$\frac{\sharp F \vdash X}{\neg F \vdash X}\,(\neg\text{L}) \qquad \frac{X \vdash \sharp F}{X \vdash \neg F}\,(\neg\text{R}) \qquad \frac{X \vdash F \quad G \vdash Y}{F \to G \vdash \sharp X \,;\, Y}\,(\to\text{L}) \qquad \frac{X \,;\, F \vdash G}{X \vdash F \to G}\,(\to\text{R})$$

Fig. 1. Proof rules for the basic display calculus $\mathcal{D}_0$

$$\frac{}{\bot_a \vdash X}\,(\bot_a\text{L}) \qquad \frac{F_i \vdash X}{F_1 \,\&_a\, F_2 \vdash X}\ i \in \{1,2\}\,(\&_a\text{L}) \qquad \frac{F \vdash X \quad G \vdash X}{F \vee_a G \vdash X}\,(\vee_a\text{L})$$

$$\frac{}{X \vdash \top_a}\,(\top_a\text{R}) \qquad \frac{X \vdash F \quad X \vdash G}{X \vdash F \,\&_a\, G}\,(\&_a\text{R}) \qquad \frac{X \vdash F_i}{X \vdash F_1 \vee_a F_2}\ i \in \{1,2\}\,(\vee_a\text{R})$$

Fig. 2. Structure-free proof rules for the "additive" logical connectives

for the *additive* logical connectives $\top_a$, $\bot_a$, $\&_a$ and $\vee_a$, and Figure 3 presents some *structural rules* governing the behaviour of the structural connectives $\emptyset$, ';' and $\sharp$. The rules in Figures 2 and 3 should be regarded as optional: if $\mathcal{D}$ is a display calculus and $\mathcal{R}$ is a list of rules from Figures 2 and 3 then the *extension* $\mathcal{D}+\mathcal{R}$ of $\mathcal{D}$ is the display calculus obtained from $\mathcal{D}$ by adding all rules in $\mathcal{R}$. We write $\mathcal{D}_0^+$ to abbreviate the extension of $\mathcal{D}_0$ with all of the structure-free rules in Figure 2.

We prove interpolation by induction over cut-free derivations, so we omit the cut rule from $\mathcal{D}_0$. The following theorem says that this omission is harmless.

$$\frac{\emptyset; X \vdash Y}{X \vdash Y}\,(\emptyset\text{C}_\text{L}) \qquad \frac{X \vdash Y; \emptyset}{X \vdash Y}\,(\emptyset\text{C}_\text{R}) \qquad \frac{X \vdash Y}{\emptyset; X \vdash Y}\,(\emptyset\text{W}_\text{L}) \qquad \frac{X \vdash Y}{X \vdash Y; \emptyset}\,(\emptyset\text{W}_\text{R})$$

$$\frac{(W; X); Y \vdash Z}{W; (X; Y) \vdash Z}\,(\alpha) \qquad \frac{X \vdash Z}{X; Y \vdash Z}\,(\text{W}) \qquad \frac{X; X \vdash Y}{X \vdash Y}\,(\text{C})$$

Fig. 3. Some structural rules

Theorem 2.6. *The following cut rule is admissible in any extension of* $\mathcal{D}_0$*:*

$$\frac{X \vdash F \quad F \vdash Y}{X \vdash Y} \text{ (Cut)}$$

Proof. (Sketch) Given the display property (Prop. 2.5), we verify that the proof rules in Figures 1–3 meet Belnap's conditions C1–C8 for cut-elimination [1]. □

Comment 2.7. *Under the formula interpretation of consecutions given by Definition 2.2, certain of our display calculi can be understood as follows:*

$\mathcal{D}_{\mathrm{MLL}} = \mathcal{D}_0 + (\alpha), (\emptyset \mathrm{C_L}), (\emptyset \mathrm{C_R}), (\emptyset \mathrm{W_L}), (\emptyset \mathrm{W_R})$ *is multiplicative linear logic (LL);*
$\mathcal{D}_{\mathrm{MALL}} = \mathcal{D}_0^+ + (\alpha), (\emptyset \mathrm{C_L}), (\emptyset \mathrm{C_R}), (\emptyset \mathrm{W_L}), (\emptyset \mathrm{W_R})$ *is multiplicative additive LL;*
$\mathcal{D}_{\mathrm{CL}} = \mathcal{D}_0 + (\alpha), (\emptyset \mathrm{C_L}), (\emptyset \mathrm{C_R}), (\mathrm{W}), (\mathrm{C})$ *is standard classical propositional logic.*

3 Interpolation: Nullary, Unary and Structure-Free Rules

We now turn to our main topic: whether *interpolation* holds in our display calculi.

Definition 3.1 (Interpolation). A display calculus $\mathcal{D}$ has the *interpolation* property if for any $\mathcal{D}$-provable consecution $X \vdash Y$ there is an *interpolant* formula I such that $X \vdash I$ and $I \vdash Y$ are both $\mathcal{D}$-provable with $\mathcal{V}(I) \subseteq \mathcal{V}(X) \cap \mathcal{V}(Y)$.

We note that, by cut-admissibility (Theorem 2.6), the existence of an interpolant for a consecution $\mathcal{C}$ implies the provability of $\mathcal{C}$.

We aim to emulate the spirit of the classical proof-theoretic approach to interpolation for cut-free sequent calculi such as Gentzen's LK (see e.g. [4]). That is, given a cut-free display calculus proof of a consecution, we aim to construct its interpolant by induction over the structure of the proof. However, the display postulates introduce a difficulty: for example, given an interpolant I for $X \,;\, Y \vdash Z$, it is not clear how to use I to obtain an interpolant for $X \vdash \sharp Y \,;\, Z$. In fact, similar problems arise for sequent calculi as well (e.g., in the classical negation rules of LK), and the usual solution is to simultaneously construct interpolants for *all possible decompositions* of each sequent. We employ an analogue of this strategy for the setting of display calculi: we simultaneously construct interpolants for *all possible rearrangements* of each consecution, where the notion of "rearrangement" is provided by the combination of display-equivalence and, if it is present in the calculus, the associativity rule (α). The latter inclusion is necessary for similar reasons to those for the inclusion of the display postulates.

Definition 3.2. Let $\mathcal{D}$ be a display calculus and $\mathcal{C}$, $\mathcal{C}'$ be consecutions. We define $\mathcal{C} \rightarrow_A \mathcal{C}'$ to hold iff $\mathcal{D}$ includes (α) and $\mathcal{C}$ is the premise of an instance of (α) with conclusion $\mathcal{C}'$. Then the relation $\rightarrow_{AD}$ is defined to be $\rightarrow_A \cup \rightleftarrows_D$ and the relation $\equiv_{AD}$ is defined to be the reflexive-transitive closure of $\rightarrow_{AD}$.

Clearly $\equiv_D \subseteq \equiv_{AD}$, and $\equiv_{AD}$ is exactly $\equiv_D$ in any display calculus without (α).

Comment 3.3. *The relation* $\equiv_{AD}$ *is indeed an equivalence relation. Furthermore, the following proof rule is derivable in any extension of* $\mathcal{D}_0$*:*

$$\frac{X' \vdash Y'}{X \vdash Y} \; X \vdash Y \equiv_{AD} X' \vdash Y' \;\; (\equiv_{AD})$$

Our definition of $\equiv_{AD}$ gives rise to the following "local AD-interpolation" property for display calculus proof rules.

Definition 3.4 (LADI property). A proof rule of a display calculus $\mathcal{D}$ with conclusion $\mathcal{C}$ is said to have the *local AD-interpolation* (LADI) property if, given that for each premise of the rule $\mathcal{C}_i$ we have interpolants for all $\mathcal{C}'_i \equiv_{AD} \mathcal{C}_i$, we can construct interpolants for all $\mathcal{C}' \equiv_{AD} \mathcal{C}$.

Lemma 3.5. *If the proof rules of a display calculus* $\mathcal{D}$ *each have the LADI property, then* $\mathcal{D}$ *has the interpolation property.*

Proof. (Sketch) We require an interpolant for each $\mathcal{D}$-provable consecution $\mathcal{C}$. We construct interpolants for all $\mathcal{C}' \equiv_{AD} \mathcal{C}$ by induction on the proof of $\mathcal{C}$, using LADI for the proof rules at each induction step, giving an interpolant for $\mathcal{C}$. □

Thus the LADI property gives a sufficient condition, in terms of individual proof rules, for interpolation to hold in display calculi. In proving this property for a given rule, we will require to track the atomic parts of a consecution being rearranged using $\equiv_{AD}$, and possibly substitute other structures for these parts. It is intuitively obvious how to do this: the next definitions formalise the concept.

Definition 3.6 (Substitution). Let Z be a part of the structure X. We write the substitution notation $X[Y/Z]$, where Y is a structure, to denote the replacement of Z (which we emphasise is a substructure *occurrence*) by the structure Y. We extend substitution to consecutions in the obvious way.

Definition 3.7 (Congruence). Let $\mathcal{C} \to_{AD} \mathcal{C}'$, whence $\mathcal{C}$ and $\mathcal{C}'$ are obtained by assigning structures to the structure variables occurring in our statement of some display postulate (see Defn. 2.4) or the rule (α) (see Figure 3). Two atomic parts A and A' of $\mathcal{C}$ and $\mathcal{C}'$ respectively are said to be *congruent* if they occupy the same position in the structure assigned to some structure variable.

(E.g., the two indicated occurrences of F are congruent in $X; (F; \emptyset) \vdash Z \to_{AD} X \vdash \sharp(F; \emptyset); Z$, as are the two indicated occurrences of $\emptyset$, because they occupy the same position in the structure $(F; \emptyset)$ assigned to the structure variable Y in our statement of the display postulate $X; Y \vdash Z \rightleftarrows_D X \vdash \sharp Y; Z$.)

We extend congruence to atomic parts A and A' of consecutions $\mathcal{C}$ and $\mathcal{C}'$ such that $\mathcal{C} \equiv_{AD} \mathcal{C}'$ by reflexive-transitive induction on $\equiv_{AD}$ in the obvious way. That is, any atomic part of $\mathcal{C}$ is congruent to itself, and if $\mathcal{C} \to_{AD} \mathcal{C}'' \equiv_{AD} \mathcal{C}'$ then A and A' are congruent if there is an atomic part A'' of $\mathcal{C}''$ such that A is congruent to A'' and A'' is congruent to A'.

Finally, we extend congruence to non-atomic parts of consecutions as follows. If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and Z, Z' are parts of $\mathcal{C}, \mathcal{C}'$ respectively then Z and Z' are congruent if every atomic part A of Z is congruent to an atomic part A' of Z', such that the position of A in Z is identical to the position of A' in Z'.

Comment 3.8. *If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ then, for any atomic part of $\mathcal{C}$, there is a unique congruent atomic part of $\mathcal{C}'$. Moreover, congruent parts of $\mathcal{C}$ and $\mathcal{C}'$ are occurrences of the same structure. We use identical names for parts of $\equiv_{AD}$-related consecutions to mean that those parts are congruent. E.g., we write $\mathcal{C}[Z/A] \equiv_{AD} \mathcal{C}'[Z/A]$ to mean that the two indicated parts A are congruent.*

Lemma 3.9 (Substitution lemma). *If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and A is an atomic part of $\mathcal{C}$ then, for any structure Z, we have $\mathcal{C}[Z/A] \equiv_{AD} \mathcal{C}'[Z/A]$.*

Proof. (Sketch) Since the display postulates and the associativity rule (α) are each closed under substitution of an arbitrary structure for congruent atomic parts, this follows by an easy reflexive-transitive induction on $\mathcal{C} \equiv_{AD} \mathcal{C}'$. □

Proposition 3.10. *The proof rules ($\equiv_D$), (Id), ($\top L$), ($\top R$), ($\bot L$), ($\bot R$), ($\neg L$), ($\neg R$), (&L), ($\vee R$), and ($\rightarrow R$) each have the LADI property in any extension of $\mathcal{D}_0$. Furthermore, the associativity rule (α) has the LADI property in any extension of $\mathcal{D}_0 + (\alpha)$, and the structure-free rules ($\top_a R$), ($\bot_a L$), ($\&_a L$), ($\&_a R$), ($\vee_a L$), and ($\vee_a R$) each have the LADI property in any extension of $\mathcal{D}_0^+$.*

Proof. (Sketch) We treat each rule separately, noting that ($\equiv_D$) and (α) are more or less immediate by assumption. We just show one case here, the structure-free rule ($\vee_a$L). In that case, we must produce interpolants for all $W \vdash Z \equiv_{AD} F \vee_a G \vdash X$. We distinguish two subcases: either the indicated $F \vee_a G$ occurs in W or in Z. We suppose it occurs in Z, so that by Lemma 3.9 we have $F \vdash X \equiv_{AD} W \vdash Z[F/F \vee_a G]$ and $G \vdash X \equiv_{AD} W \vdash Z[G/F \vee_a G]$. Let I_1 and I_2 be the interpolants given by assumption for $W \vdash Z[F/F \vee_a G]$ and $W \vdash Z[G/F \vee_a G]$ respectively. We claim that $I_1 \&_a I_2$ is an interpolant[1] for $W \vdash Z$. The variable condition is easily seen to hold, so it remains to check the provability conditions. Given that $W \vdash I_1$ and $W \vdash I_2$ are provable by assumption, we can derive $W \vdash I_1 \&_a I_2$ by a single application of ($\&_a$R). Finally, given that $I_1 \vdash Z[F/F \vee_a G]$ and $I_2 \vdash Z[G/F \vee_a G]$ are provable by assumption, we must show that $I_1 \&_a I_2 \vdash Z$ is provable. First, since the indicated $F \vee_a G$ occurs in Z by assumption, we have $I_1 \&_a I_2 \vdash Z \equiv_D F \vee_a G \vdash U$ for some U by the display property (Prop. 2.5). Thus by Lemma 3.9 we have $I_1 \&_a I_2 \vdash Z[F/F \vee_a G] \equiv_D F \vdash U$ and $I_1 \&_a I_2 \vdash Z[G/F \vee_a G] \equiv_D G \vdash U$. So we can derive $I_1 \&_a I_2 \vdash Z$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{I_1 \vdash Z[F/F \vee_a G]}}{I_1 \&_a I_2 \vdash Z[F/F \vee_a G]}(\&_a\text{L})}{F \vdash U}(\equiv_D) \qquad \dfrac{\dfrac{\dfrac{\vdots}{I_2 \vdash Z[G/F \vee_a G]}}{I_1 \&_a I_2 \vdash Z[G/F \vee_a G]}(\&_a\text{L})}{G \vdash U}(\equiv_D)}{\dfrac{F \vee_a G \vdash U}{I_1 \&_a I_2 \vdash Z}(\equiv_D)}(\vee_a\text{L})$$

If the indicated $F \vee_a G$ instead occurs in W, then the argument is similar but we pick the interpolant to be $I_1 \vee_a I_2$. This completes the case, and the proof. □

[1] Equivalently, $\neg(\neg I_1 \vee_a \neg I_2)$ also works. Note that, because of the display postulate $X \vdash Y \rightleftarrows_D \sharp Y \vdash \sharp X$, it is not possible to construct $\neg$-free interpolants in general.

4 Interpolation: Binary Logical Rules

We now extend our basic method for proving LADI of display calculus proof rules to the binary logical rules of $\mathcal{D}_0$. These cases are considerably harder than the simple rules considered in the previous section because they combine arbitrary structures from the two premises, leading to many new $\equiv_{AD}$-rearrangements of the conclusion compared to the premises. To deal with this complexity, we will require several technical substitutivity lemmas for $\equiv_{AD}$.

The following notion of *deletion* of a part of a structure is similar to that used by Restall [17]. We write $\sharp^n$ for a string of n occurrences of $\sharp$. Recall that identical names are used to denote congruent parts of $\equiv_{AD}$-equivalent consecutions.

Definition 4.1 (Deletion). A part Z of a structure X is *delible from* X if X is not of the form $\sharp^n Z$ for some $n \geq 0$, i.e., X contains a substructure occurrence of the form $\sharp^n Z; W$ (up to commutativity of ";"). If Z is delible from X, we write $X \setminus Z$ for the structure $X[W/(\sharp^n Z; W)]$, the result of deleting Z from X.

A part Z of a consecution $\mathcal{C}$ is delible from $\mathcal{C}$ if it can be deleted from the side of $\mathcal{C}$ of which it is a part, and we write $\mathcal{C} \setminus Z$ for the consecution obtained by deleting Z from the appropriate side of $\mathcal{C}$.

The following lemma says that $\equiv_{AD}$-rearrangement is (essentially) preserved under deletion of congruent parts. This is crucial to the subsequent substitutivity Lemmas 4.3 and 4.5, which say that $\equiv_{AD}$-rearrangement does not depend on the presence of "contextual" structure not directly affected by the rearrangement.

Lemma 4.2 (Deletion lemma). *Let $\mathcal{C}$ be a consecution and let A be an atomic part of $\mathcal{C}$. If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and A is delible from $\mathcal{C}$ then the following hold:*

1. *if A is delible from $\mathcal{C}'$ then $\mathcal{C} \setminus A \equiv_{AD} \mathcal{C}' \setminus A$;*
2. *if A is not delible from $\mathcal{C}'$ then one side of $\mathcal{C}'$ is of the form $\sharp^m(Z_1; Z_2)$ and we have $\mathcal{C} \setminus A \equiv_{AD} Z_1 \vdash \sharp Z_2$ if $(Z_1; Z_2)$ is an antecedent part of $\mathcal{C}'$, and $\mathcal{C} \setminus A \equiv_{AD} \sharp Z_1 \vdash Z_2$ if $(Z_1; Z_2)$ is a consequent part of $\mathcal{C}'$.*

Proof. (Sketch) By reflexive-transitive induction on $\mathcal{C} \equiv_{AD} \mathcal{C}'$. In the reflexive case we have $\mathcal{C}' = \mathcal{C}$ and are trivially done. In the transitive case we have $\mathcal{C} \equiv_{AD} \mathcal{C}'' \rightarrow_{AD} \mathcal{C}'$, and we distinguish subcases on $\mathcal{C}'' \rightarrow_{AD} \mathcal{C}'$. The nonstraightforward subcases are those where A is delible from $\mathcal{C}''$ but not from $\mathcal{C}'$ or vice versa. For example, consider the case $S; T \vdash U \rightarrow_{AD} S \vdash \sharp T; U$, and suppose (for 1) that A is delible from $S \vdash \sharp T; U$ but not from $S; T \vdash U$. Then we must have $U = \sharp^n A$, whence we have by part 2 of the induction hypothesis that $(S; T \vdash U) \setminus A \equiv_{AD} S \vdash \sharp T$ (because $S; T$ is an antecedent part of $S; T \vdash U$). Then 1 holds as required because, given $U = \sharp^n A$, we have $S \vdash \sharp T = (S \vdash \sharp T; U) \setminus A$. □

Lemma 4.3 (Substitutivity I). *For all W, X, Y, Z, if $W \vdash X \equiv_{AD} W \vdash Y$ then $Z \vdash X \equiv_{AD} Z \vdash Y$, and if $X \vdash W \equiv_{AD} Y \vdash W$ then $X \vdash Z \equiv_{AD} Y \vdash Z$.*

Proof. (Sketch) By Lemma 3.9 it suffices to consider the case in which Z is a formula F. We prove both implications simultaneously by structural induction on W. The atomic case follows by Lemma 3.9. The case $W = \sharp W'$ is straightforward by induction hypothesis. In the case $W = W_1; W_2$ we obtain, for the first implication, $F; G \vdash X \equiv_{AD} F; G \vdash Y$ using the induction hypothesis. Thus we obtain, by Lemma 4.2, $(F; G \vdash X) \setminus G \equiv_{AD} (F; G \vdash Y) \setminus G$, i.e. $F \vdash X = F \vdash Y$ as required. The second implication is similar. □

Definition 4.4. *Let $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and let Z, Z' be parts of $\mathcal{C}$ and $\mathcal{C}'$ respectively. We write $Z' \lhd Z$ if every atomic part of Z' is congruent to an atomic part of Z.*

Lemma 4.5 (Substitutivity II). *For all structures W, W', X, Y and for any atomic structure A, all of the following hold:*

1. *if $W \vdash X \equiv_{AD} W' \vdash Y$ and $W' \lhd W$ then $\exists U.\ W \vdash A \equiv_{AD} W' \vdash U$;*
2. *if $X \vdash W \equiv_{AD} W' \vdash Y$ and $W' \lhd W$ then $\exists U.\ A \vdash W \equiv_{AD} W' \vdash U$;*
3. *if $W \vdash X \equiv_{AD} Y \vdash W'$ and $W' \lhd W$ then $\exists U.\ W \vdash A \equiv_{AD} U \vdash W'$;*
4. *if $X \vdash W \equiv_{AD} Y \vdash W'$ and $W' \lhd W$ then $\exists U.\ A \vdash W \equiv_{AD} U \vdash W'$.*

(Also, in each case we still have $W' \lhd W$ under the replacement of X by A.)

Proof. (Sketch) We show all four implications simultaneously by structural induction on X. The atomic case follows from Lemma 3.9. The case $X = \sharp X'$ is straightforward by induction hypothesis. In the case $X = X_1; X_2$ we obtain, for the first implication, $W \vdash A; A \equiv_{AD} W' \vdash V$ for some V using the induction hypothesis (twice). Since $W' \lhd W$, both indicated occurrences of A must occur in, and be delible from V. Thus by Lemma 4.2, we have $W \vdash A \equiv_{AD} W' \vdash (V \setminus A)$ and are done by taking $U = V \setminus A$. The other implications are similar. □

Our final lemma says that if two separate structures have been "mixed up" by $\equiv_{AD}$, then the resulting structure can be "filtered" into its component parts.

Lemma 4.6 (Filtration). *Let $X; Y \vdash U \equiv_{AD} W \vdash Z$, where $W \lhd X; Y$ but $W \not\lhd X$ and $W \not\lhd Y$. Then there exist W_1 and W_2 such that $W \vdash Z \equiv_{AD} W_1; W_2 \vdash Z$ with $W_1 \lhd X$ and $W_2 \lhd Y$. Similarly, if $X; Y \vdash U \equiv_{AD} Z \vdash W$ with $W \lhd X; Y$ but $W \not\lhd X$ and $W \not\lhd Y$, then there exist W_1 and W_2 such that $Z \vdash W \equiv_{AD} Z \vdash W_1; W_2$ with $W_1 \lhd X$ and $W_2 \lhd Y$.*

Proof. (Sketch) We prove both implications simultaneously by structural induction on W. The difficult case is when $W = W_1; W_2$. If $W_1 \lhd X$ and $W_2 \lhd Y$ or vice versa then we are done. If not, in the case of the first implication we have $X; Y \vdash U \equiv_{AD} W_1; W_2 \vdash Z$ where $W_1; W_2 \lhd X; Y$ and either $W_1 \not\lhd X$ and $W_1 \not\lhd Y$, or $W_2 \not\lhd X$ and $W_2 \not\lhd Y$ (we assume both here). It is clear by inspection of the display postulates that this situation can only arise when the rule (α) is present. Using the induction hypotheses, we obtain $W_1' \lhd X$, $W_1'' \lhd Y$, $W_2' \lhd X$ and $W_2'' \lhd Y$ such that $W_1; W_2 \vdash Z \equiv_{AD} (W_1'; W_1''); (W_2'; W_2'') \vdash Z$. Thus, given that $\equiv_{AD}$ incorporates (α), we obtain $W_1; W_2 \vdash Z \equiv_{AD} (W_1'; W_2'); (W_1''; W_2'') \vdash Z$ where $(W_1'; W_2') \lhd X$ and $(W_1''; W_2'') \lhd Y$ as required. □

Theorem 4.7 (Binary rules). *The rules $(\&R)$, $(\vee L)$ and $(\rightarrow L)$ all have the local AD-interpolation property in any extension of $\mathcal{D}_0$.*

Proof. (Sketch) We consider the case (&R), in which case we must produce interpolants for all $W \vdash Z \equiv_{AD} X;Y \vdash F\&G$. We suppose the indicated $F\&G$ occurs in Z, in which case $W \lhd X;Y$, and distinguish three subcases: $W \lhd X$; $W \lhd Y$; and $W \ntriangleleft X$, $W \ntriangleleft Y$. We just show the last case, the hardest. By the first part of Lemma 4.6 there exist W_1 and W_2 such that $W \vdash Z \equiv_{AD} W_1;W_2 \vdash Z$ with $W_1 \lhd X$ and $W_2 \lhd Y$. Thus we have $X \vdash \sharp Y; F \;\&\; G \equiv_{AD} W_1 \vdash \sharp W_2; Z$ with $W_1 \lhd X$, and $Y \vdash \sharp X; F \;\&\; G \equiv_{AD} W_2 \vdash \sharp W_1; Z$ with $W_2 \lhd Y$. Hence by part 1 of Lemma 4.5 we have $X \vdash F \equiv_{AD} W_1 \vdash U_1$ for some U_1 and $Y \vdash G \equiv_{AD} W_2 \vdash U_2$ for some U_2. Let I_1, I_2 be the interpolants given by assumption for $W_1 \vdash U_1$ and $W_2 \vdash U_2$ respectively. We claim that $I_1 \;\&\; I_2$ is an interpolant for $W \vdash Z$.

First, we show that $W \vdash I_1 \;\&\; I_2$ is provable. We have $W_1 \vdash I_1$ and $W_2 \vdash I_2$ provable by assumption, so $W_1;W_2 \vdash I_1 \;\&\; I_2$ is provable by applying (&R). Since $W_1;W_2 \vdash Z \equiv_{AD} W \vdash Z$, we have $W_1;W_2 \vdash I_1 \;\&\; I_2 \equiv_{AD} W \vdash I_1 \;\&\; I_2$ by Lemma 4.3, and so $W \vdash I_1 \;\&\; I_2$ is provable by applying the rule ($\equiv_{AD}$).

Next, we must show that $I_1 \;\&\; I_2 \vdash Z$ is derivable, given that $I_1 \vdash U_1$ and $I_2 \vdash U_2$ are derivable. First, note that because $X \vdash F \equiv_{AD} W_1 \vdash U_1$ and $W_1 \lhd X$, the indicated F is a part of U_1, and thus $I_1 \vdash U_1 \equiv_D V_1 \vdash F$ for some V_1 by Prop. 2.5. Similarly, $I_2 \vdash U_2 \equiv_D V_2 \vdash G$ for some V_2. Next, using Lemma 3.9, we have $W_1 \vdash \sharp W_2; Z \equiv_{AD} W_1 \vdash U_1[(\sharp Y; F \;\&\; G)/F]$. Thus by Lemma 4.3 we have $I_1 \vdash \sharp W_2; Z \equiv_{AD} I_1 \vdash U_1[(\sharp Y; F \;\&\; G)/F]$. Since $I_1 \vdash U_1 \equiv_D V_1 \vdash F$ we have, using Lemma 3.9, $I_1 \vdash \sharp W_2; Z \equiv_{AD} V_1; Y \vdash F \;\&\; G$. Now, since $Y \vdash G \equiv_{AD} W_2 \vdash U_2$ we obtain using Lemma 3.9 $W_2 \vdash \sharp I_1; Z \equiv_{AD} W_2 \vdash U_2[(\sharp V_1; F \;\&\; G)/G]$. So by applying Lemma 4.3 we have $I_2 \vdash \sharp I_1; Z \equiv_{AD} I_2 \vdash U_2[\sharp V_1; F \;\&\; G/G]$. Since $I_2 \vdash U_2 \equiv_D V_2 \vdash G$ we obtain $I_1; I_2 \vdash Z \equiv_{AD} V_1; V_2 \vdash F \;\&\; G$ (again using Lemma 3.9). This enables us to derive $I_1 \;\&\; I_2 \vdash Z$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{I_1 \vdash U_1}}{V_1 \vdash F}(\equiv_D) \qquad \dfrac{\dfrac{\vdots}{I_2 \vdash U_2}}{V_2 \vdash G}(\equiv_D)}{\dfrac{V_1; V_2 \vdash F \;\&\; G}{\dfrac{I_1; I_2 \vdash Z}{I_1 \;\&\; I_2 \vdash Z}(\&L)}(\equiv_{AD})}(\&R)$$

Finally, we check the variable condition. We have $\mathcal{V}(I_1) \subseteq \mathcal{V}(W_1) \cap \mathcal{V}(U_1)$ and $\mathcal{V}(I_2) \subseteq \mathcal{V}(W_2) \cap \mathcal{V}(U_2)$. It is clear that $\mathcal{V}(W_1) \subseteq \mathcal{V}(W)$ and $\mathcal{V}(W_2) \subseteq \mathcal{V}(W)$ because $W \vdash Z \equiv_{AD} W_1; W_2 \vdash Z$. Moreover, $\mathcal{V}(U_1) \subseteq \mathcal{V}(Z)$ because we have $X \vdash F \equiv_{AD} W_1 \vdash U_1$ and $X \vdash \sharp Y; F \;\&\; G \equiv_{AD} W_1 \vdash \sharp W_2; Z$ while $W_1 \lhd X$ and $W_2 \lhd Y$ (or, alternatively, it is clear by inspection of the derivation above). Similarly $\mathcal{V}(U_2) \subseteq \mathcal{V}(Z)$ and thus $\mathcal{V}(I_1 \;\&\; I_2) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$ as required.

The subcases $W \lhd X$ and $W \lhd Y$ are similar except that we directly use the interpolant given by just one of the premises. If the indicated $F\&G$ occurs in W rather than Z we again distinguish three subcases and take the interpolant $I_1 \vee I_2$ in the analogue of the subcase above. This completes the proof. □

Corollary 4.8. *For any $\mathcal{D} \in \{\mathcal{D}_0, \mathcal{D}_0^+, \mathcal{D}_0 + (\alpha), \mathcal{D}_0^+ + (\alpha)\}$, the proof rules of $\mathcal{D}$ all have the LADI property in (any extension of) $\mathcal{D}$, and thus $\mathcal{D}$ has the interpolation property.*

Proof. LADI for the proof rules of $\mathcal{D}$ in any extension of $\mathcal{D}$ is given by Prop. 3.10 and Theorem 4.7. Interpolation for $\mathcal{D}$ then follows by Lemma 3.5. □

5 Interpolation: Structural Rules

We now examine the LADI property for the structural rules given in Figure 3.

Proposition 5.1 (Unit contraction rules). *The unit left-contraction rule $(\emptyset C_L)$ has the LADI property in any extension of $\mathcal{D}_0 + (\emptyset C_L)$. Similarly, the rule $(\emptyset C_R)$ has the LADI property in any extension of $\mathcal{D}_0 + (\emptyset C_R)$.*

Proof. (Sketch) We consider $(\emptyset C_L)$ here; $(\emptyset C_R)$ is similar. We require to construct interpolants for all $W \vdash Z \equiv_{AD} X \vdash Y$. First, by reflexive-transitive induction on $W \vdash Z \equiv_{AD} X \vdash Y$, we show that $\emptyset; X \vdash Y \equiv_{AD} (W \vdash Z)[(\emptyset; U)/U]$ or $\emptyset; X \vdash Y \equiv_{AD} (W \vdash Z)[(\sharp\emptyset; U)/U]$ for some U. We claim that the assumed interpolant I for $W' \vdash Z'$ is an interpolant for $W \vdash Z$. The variable condition is easily seen to be satisfied, so it remains to check the provability conditions. We assume without loss of generality that U is a part of Z, so that $W \vdash I$ is provable by assumption. To prove $I \vdash Z$, we start with the assumed derivation of $I \vdash Z'$ and use Prop. 2.5 to display the structure $\emptyset; U$ or $\sharp\emptyset; U$. We then remove the $\emptyset$ using $(\emptyset C_L)$ and obtain $I \vdash Z$ by inverting the previous display moves. □

Proposition 5.2 (Unit weakening rules). *The unit weakening rule $(\emptyset W_L)$ has the LADI property in any extension of $\mathcal{D}_0 + (\emptyset W_L)$. Similarly, the rule $(\emptyset W_R)$ has the LADI property in any extension of $\mathcal{D}_0 + (\emptyset W_R)$.*

Proof. (Sketch) We just consider $(\emptyset W_L)$, as $(\emptyset W_R)$ is similar. We require to find interpolants for all $W \vdash Z \equiv_{AD} \emptyset; X \vdash Y$. We distinguish two cases. First of all, if the indicated $\emptyset$ is not delible from $W \vdash Z$, then W or Z is of the form $\sharp^n\emptyset$. We suppose $Z = \sharp^n\emptyset$ in which case n must be odd (because the indicated $\emptyset$ is an antecedent part of $\emptyset; X \vdash Y$ and thus of $W \vdash Z$) and we pick the interpolant for $W \vdash Z$ to be $\neg\top$. The variable condition is trivially satisfied, and $\neg\top \vdash Z = \neg\top \vdash \sharp^n\emptyset$ is easily provable. $W \vdash \neg\top$ is provable from the premise $X \vdash Y$ using the rule $(\emptyset W_L)$ and the derived rule $(\equiv_{AD})$ by observing that $\emptyset; X \vdash Y \equiv_{AD} \emptyset \vdash \sharp W$. The case where $W = \sharp^n\emptyset$ is symmetric.

If the indicated $\emptyset$ *is* delible from $W \vdash Z$ then, by Lemma 4.2, we have $X \vdash Y = (\emptyset; X \vdash Y) \setminus \emptyset \equiv_{AD} (W \vdash Z) \setminus \emptyset$. We claim that the interpolant I given for $(W \vdash Z) \setminus \emptyset$ by assumption is also an interpolant for $W \vdash Z$. Without loss of generality, we assume that the indicated $\emptyset$ occurs in Z, so that $(W \vdash Z) \setminus \emptyset = W \vdash (Z \setminus \emptyset)$. It is easy to see that the required variable condition holds. It remains to check the provability conditions. We have $W \vdash I$ provable by assumption, so it just remains to show that $I \vdash Z$ is provable, given that $I \vdash (Z \setminus \emptyset)$ is provable. By the definition of deletion (Defn. 4.1), $Z \setminus \emptyset = Z[U/(\sharp^n\emptyset; U)]$ for some U. Thus, by Prop. 2.5, and assuming $(\sharp^n\emptyset; U)$ an antecedent part of Z, we have $I \vdash Z \equiv_D \emptyset; U \vdash V$ and $I \vdash (Z \setminus \emptyset) \equiv_D U \vdash V$ for some V (note that the *same* V is obtained in both cases). Thus we can derive $I \vdash Z$ from $I \vdash (Z \setminus \emptyset)$ by applications of $(\equiv_D)$ and $(\emptyset \mathrm{W_L})$. □

Theorem 5.3 (Weakening). *The weakening rule* (W) *has the LADI property in any extension of* $\mathcal{D}_0 + (\mathrm{W}), (\emptyset \mathrm{C_L})$ *or* $\mathcal{D}_0 + (\mathrm{W}), (\emptyset \mathrm{C_R})$.

Proof. (Sketch) We require to find interpolants for all $W \vdash Z \equiv_{AD} X; X' \vdash Y$. We distinguish three cases: $W \lhd X'$; $Z \lhd X'$; and $W \ntriangleleft X'$, $Z \ntriangleleft X'$. In the case $W \lhd X'$, we choose the interpolant I to be $\top$ if $(\emptyset \mathrm{C_L})$ is available (which guarantees $W \vdash \top$ is provable), or $\neg\bot$ if $(\emptyset \mathrm{C_R})$ is available (which guarantees $W \vdash \neg\bot$ is provable). To see that $I \vdash Z$ is provable, note that $W \vdash Z \equiv_{AD} X' \vdash \sharp X; Y$ with $(\sharp X; Y) \lhd Z$. Thus, using part 4 of Lemma 4.5, we have $I \vdash Z \equiv_{AD} X; U \vdash Y$ for some U, whence we can derive $I \vdash Z$ from the premise $X \vdash Y$ by applying (W) and the derived rule $(\equiv_{AD})$. The case $Z \lhd X'$ is symmetric. In the case $W \ntriangleleft X'$ and $Z \ntriangleleft X'$, we first show that there are atomic parts $A_1, \ldots, A_n$ of X' with

$$X \vdash Y \equiv_{AD} (\ldots(((W \vdash Z) \setminus A_1) \setminus A_2) \ldots) \setminus A_n = W' \vdash Z'$$

(This can be proven by structural induction on X', using Lemma 4.2 in the atomic case.) We claim that the interpolant I for $W' \vdash Z'$ given by assumption is also an interpolant for $W \vdash Z$. First we check the variable condition. We have $\mathcal{V}(I) \subseteq \mathcal{V}(W') \cap \mathcal{V}(Z')$ by assumption. It is clear that $\mathcal{V}(W') \subseteq \mathcal{V}(W)$ and $\mathcal{V}(Z') \subseteq \mathcal{V}(Z)$ since W' and Z' are obtained by deleting some parts of W and Z respectively. Thus $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$ as required.

It remains to check the provability conditions. We have $W' \vdash I$ provable by assumption. By the definition of deletion (Defn. 4.1), W' is obtained from W by replacing a number of substructure occurrences of the form $\sharp^n A; S$ by S. We obtain the required derivation of $W \vdash I$ by, working backwards, using the display property (Prop. 2.5) to display each such $\sharp^n A; S$ and then removing $\sharp^n A$ using (W). (Formally, we proceed by induction on the number of substructure occurrences deleted from W to obtain W'.) Deriving $I \vdash Z$ is similar. □

Proposition 5.4 (Contraction). *The contraction rule* (C) *has the LADI property in any extension of* $\mathcal{D}_0 + (\alpha)$.

Proof. (Sketch) We require to find interpolants for all $W \vdash Z \equiv_{AD} X \vdash Y$. First, using the fact that $\equiv_{AD}$ contains (α) by assumption, we show that there exist atomic parts $A_1, \ldots, A_n$ of X such that

$$X; X \vdash Y \equiv_{AD} (W \vdash Z)[(A_1; A_1)/A_1, \ldots, (A_n; A_n)/A_n] = W' \vdash Z'$$

(This is proven by structural induction on X, using Lemma 3.9 in the atomic case.) We claim that the interpolant I for $W' \vdash Z'$ given by assumption is also an interpolant for $W \vdash Z$. We have $\mathcal{V}(I) \subseteq \mathcal{V}(W') \cap \mathcal{V}(Z')$ by assumption and, clearly, $\mathcal{V}(W) = \mathcal{V}(W')$ and $\mathcal{V}(Z) = \mathcal{V}(Z')$, so we easily have the required variable condition $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$.

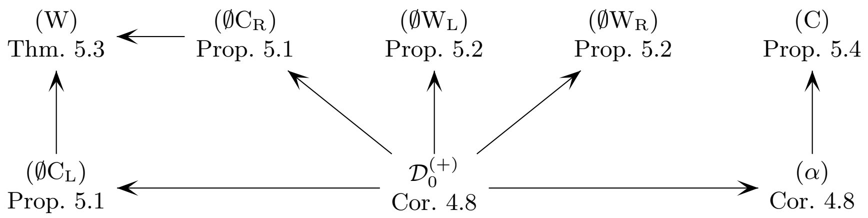

Fig. 4. Diagrammatic summary of our results. Local AD-interpolation of the proof rule(s) at a node holds in a calculus with all of the proof rules at its ancestor nodes.

Next we check the provability conditions. We have $W' \vdash I$ provable by assumption, and W' is obtained from W by replacing a number of its atomic parts A by the structure $A; A$. We obtain the required derivation of $W \vdash I$ by, working backwards, using the display property (Prop. 2.5) to display each such A and then duplicating it using (C). (Formally, we proceed by induction on the number of atomic parts of W duplicated to obtain W'.) Deriving $I \vdash Z$ is similar. □

Our reliance on the presence of the associativity rule (α) in Prop. 5.4 can be motivated by considering the following instance of contraction:

$$\frac{(X_1; X_2); (X_1; X_2) \vdash Y}{X_1; X_2 \vdash Y}$$

For the LADI property, we need in particular an interpolant for $X_1 \vdash \sharp X_2; Y \equiv_D X_1; X_2 \vdash Y$. However, without associativity, we cannot rearrange the premise into $X_1; X_1 \vdash (\sharp X_2; \sharp X_2); Y$ as would otherwise be provided by Prop. 5.4. The best we can do without associativity is $X_1 \vdash \sharp X_2; (\sharp(X_1; X_2); Y)$, whose interpolant I is too weak to serve as an interpolant for $X_1 \vdash \sharp X_2; Y$ both in terms of provability and in terms of the variable condition. A similar problem occurs if there is more than one binary structural connective, even if both are associative.

The various conditions for the LADI property to hold of each proof rule are set out in Figure 4. In consequence, we have the following interpolation results.

Theorem 5.5 (Interpolation). *Let $\mathcal{D}$ be an extension of $\mathcal{D}_0$ where: if $\mathcal{D}$ contains* (C) *it must also contain* (α)*, and if $\mathcal{D}$ contains* (W) *then it must also contain either* ($\emptyset C_L$) *or* ($\emptyset C_R$)*. Then $\mathcal{D}$ has the interpolation property.*

Proof. By Lemma 3.5 it suffices to prove the LADI property in $\mathcal{D}$ for each proof rule of $\mathcal{D}$. The rules of $\mathcal{D}_0$, and (α) if applicable, satisfy the LADI property in $\mathcal{D}$ by Corollary 4.8. The other structural rules of $\mathcal{D}$, if applicable, satisfy LADI in $\mathcal{D}$ by Theorem 5.3 and Propositions 5.1, 5.2 and 5.4.

Drawing on the observations in Comment 2.7, Thm 5.5 yields the following:

Corollary 5.6. *$\mathcal{D}_{\mathrm{MLL}}$, $\mathcal{D}_{\mathrm{MALL}}$ and $\mathcal{D}_{\mathrm{CL}}$ all have the interpolation property.*

6 Related and Future Work

Our central contribution is a general, fully constructive proof-theoretic method for proving Craig interpolation in a large class of displayable logics, based upon an analysis of the individual rules of the display calculi. This analysis is "as local as possible" in that the LADI property required for each proof rule typically depends only on the presence of certain other rules in the calculus, and the syntax of the rule itself. The practicality and generality of our method is demonstrated by its application to a fairly large family of display calculi differing in their structural rules (and the presence or otherwise of additive logical connectives). We obtain by this uniform method the interpolation property for MLL, MALL and ordinary classical logic, as well as numerous variants of these logics. To our knowledge, ours are the first interpolation results based on display calculi, thereby answering positively Belnap's long-standing open question (see p1) about this possibility.

While interpolation based on display calculi appears to be new, interpolation for substructural logics is of course not new. The closest work to ours is probably Roorda's on interpolation for various fragments of classical linear logic [18], using induction over cut-free sequent calculus proofs. Roorda also identifies fragments where interpolation fails (usually because certain logical connectives are unavailable). Many of Roorda's positive interpolation results overlap with our own but we cover some additional logics (e.g., nonassociative, strict or affine variants, plus full classical logic) and offer an analysis of the roles played by individual structural rules. An entirely different approach to interpolation for substructural logics is offered by Galatos and Ono [9], who establish very general interpolation theorems for certain substructural logics extending the Lambek calculus, based on their algebraisations.

Our methodology transfers easily to calculi for intuitionistic logics in which our "classical" display postulates (Defn. 2.4) are replaced by "residuated" ones of the form $X, Y \vdash Z \rightleftarrows_D X \vdash Y, Z \rightleftarrows_D Y, X \vdash Z$ (where the comma is interpreted as conjunction in antecedent position and as implication in consequent position). A more challenging technical extension is to the case where we have such a family of structural connectives *alongside* the first, as is typically needed to display relevant logics [17] or bunched logics [2]. Here, the main technical obstacle is in extending the substitutivity principles in Section 4 to the more complex notion of display-equivalence induced by this extension. Other possible extensions to our calculi include the addition of modalities, quantifiers or linear exponentials. In the main, these extensions appear more straightforward than adding new connective families, since they necessitate little or no modification to display-equivalence. We also note that our notion of interpolant in this paper is relatively blunt since it does not distinguish between positive and negative occurrences of variables. It should be possible to read off a sharpened version of interpolation, that does make this distinction, more or less directly from our proof.

As well as showing interpolation for a variety of substructural logics, our proof gives insights into the reasons why interpolation fails in some logics. Specifically, we identify contraction as being just as problematic for interpolation as it typically is for decidability (and even weakening causes an issue for interpolation when the logic lacks strong units). Our interpolation method is bound to fail for any multiple-family display calculus including a contraction rule, due to our observation that contraction generally has the required LADI property only in circumstances which are precluded by the presence of multiple binary structural connectives. This observation is in keeping with the fact that interpolation fails for the relevant logic **R**, as shown by Urquhart [19], since its display calculus employs two families of connectives and a contraction rule. We conjecture that interpolation might fail in bunched logics such as BI for similar reasons.

The technical overhead of our method is fairly substantial, but the techniques themselves are elementary: we mainly appeal to structural and reflexive-transitive inductions. This means that our proofs are good candidates for mechanisation in a theorem proving assistant. Dawson is currently working on an Isabelle formalisation of our proofs, based upon earlier work on mechanising display calculus with Goré [7]. As well as providing the greatest possible degree of confidence in our proofs, such a mechanisation might eventually provide the basis for an automated interpolation tool.

References

1. Belnap Jr., N.D.: Display logic. Journal of Philosophical Logic 11, 375–417 (1982)
2. Brotherston, J.: A unified display proof theory for bunched logic. In: Proceedings of MFPS-26. ENTCS, pp. 197–211. Elsevier, Amsterdam (2010)
3. Brotherston, J., Goré, R.: Craig interpolation in displayable logics. Tech. Rep. DTR11-1, Imperial College London (2011)
4. Buss, S.R.: Introduction to Proof Theory. In: Handbook of Proof Theory, ch. I. Elsevier Science, Amsterdam (1998)
5. Caniart, N.: MERIT: An interpolating model-checker. In: Touili, T., Cook, B., Jackson, P. (eds.) CAV 2010. LNCS, vol. 6174, pp. 162–166. Springer, Heidelberg (2010)
6. Craig, W.: Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. Journal of Symbolic Logic 22(3), 269–285 (1957)
7. Dawson, J.E., Goré, R.: Formalised cut admissibility for display logic. In: Carreño, V.A., Muñoz, C.A., Tahar, S. (eds.) TPHOLs 2002. LNCS, vol. 2410, pp. 131–147. Springer, Heidelberg (2002)
8. Feferman, S.: Harmonious logic: Craig's interpolation theorem and its descendants. Synthese 164, 341–357 (2008)
9. Galatos, N., Ono, H.: Algebraization, parametrized local deduction theorem and interpolation for substructural logics over FL. Studia Logica 83, 279–308 (2006)
10. Goré, R.: Gaggles, Gentzen and Galois: How to display your favourite substructural logic. Logic Journal of the IGPL 6(5), 669–694 (1998)
11. Goré, R.: Substructural logics on display. Logic J of the IGPL 6(3), 451–504 (1998)
12. Jhala, R., Majumdar, R., Xu, R.-G.: State of the union: Type inference via craig interpolation. In: Grumberg, O., Huth, M. (eds.) TACAS 2007. LNCS, vol. 4424, pp. 553–567. Springer, Heidelberg (2007)
13. Konev, B., Lutz, C., Ponomaryov, D., Wolter, F.: Decomposing description logic ontologies. In: 12th Knowledge Representation and Reasoning Conf. AAAI, Menlo Park (2010)

14. Kracht, M.: Power and weakness of the modal display calculus. In: Wansing, H. (ed.) Proof Theory of Modal Logic, pp. 93–121. Kluwer Academic Publishers, Boston (1996)
15. McMillan, K.L.: Applications of craig interpolants in model checking. In: Halbwachs, N., Zuck, L.D. (eds.) TACAS 2005. LNCS, vol. 3440, pp. 1–12. Springer, Heidelberg (2005)
16. McMillan, K.L.: Quantified invariant generation using an interpolating saturation prover. In: Ramakrishnan, C.R., Rehof, J. (eds.) TACAS 2008. LNCS, vol. 4963, pp. 413–427. Springer, Heidelberg (2008)
17. Restall, G.: Displaying and deciding substructural logics 1: Logics with contraposition. Journal of Philosophical Logic 27, 179–216 (1998)
18. Roorda, D.: Interpolation in fragments of classical linear logic. Journal of Symbolic Logic 59(2), 419–444 (1994)
19. Urquhart, A.: Failure of interpolation in relevant logics. Journal of Philosophical Logic 22, 449–479 (1993)
20. Wansing, H.: Displaying Modal Logics. Kluwer, Boston (1998)