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Bunched Logics Displayed

Abstract. We formulate a unified display calculus proof theory for the four principal varieties of bunched logic by combining display calculi for their component logics. Our calculi satisfy cut-elimination, and are sound and complete with respect to their standard presentations. We show how to constrain applications of display-equivalence in our calculi in such a way that an exhaustive proof search need be only finitely branching, and establish a full deduction theorem for the bunched logics with classical additives, BBI and CBI. We also show that the standard sequent calculus for BI can be seen as a reformulation of its display calculus, and argue that analogous sequent calculi for the other varieties of bunched logic are very unlikely to exist.

Keywords: bunched logic, display calculus, proof theory, cut-elimination

1. Introduction

Bunched logics, originating in O'Hearn and Pym's BI [22], constitute a relatively recent addition to the *menagerie* of substructural logics with practical importance in computer science. Of their better-established cousins, bunched logics most resemble *relevant logics* [25] in that they feature both *multiplicative* (a.k.a. 'intensional') and *additive* (a.k.a. 'extensional') logical connectives, with the difference between the two types characterised as a matter of which structural principles are admitted by each. However, while in relevant logics certain of the additive connectives are barred in order to exclude the paradoxes of material implication and other 'relevance-breaking' principles, in bunched logics one simply takes a full set of additive connectives as equal partners alongside the multiplicatives. Thus bunched logics can be seen as the result of freely combining an ordinary propositional logic with a multiplicative fragment of linear logic. These simple-minded constructions give rise to a *resource interpretation* of bunched logics via their Kripke semantics: formulas are interpreted as sets of resources, with the additive connectives having their standard propositional meanings and the multiplicatives, roughly speaking, denoting resource composition properties [24]. In computer science, such resource readings of bunched logic have very successfully been exploited to obtain customised logics for program

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analysis. Most notably, *separation logic* [28] — a Hoare-style framework that interprets bunched logic in models of heap memory — has spawned a host of program analysis applications that discover and reason about the structure of heap memory during program execution (recent examples include [10, 11, 13]). Bunched logic has also been variously employed in addressing other computing problems such as polymorphic abstraction [12], tree update [9], typed reference update and disposal [3] and informational dependence and independence [1].

In this paper, we examine bunched logic from the general proof-theoretic perspective. While there has been considerable interest in the semantics of bunched logics, owing mainly to the computational significance of the resulting models [15, 14, 17, 24], their proof theory by contrast has received comparatively little attention. As observed by Pym [23], it is natural to consider four principal varieties of bunched logic, characterised by the presence or otherwise of classical negation in the additive and multiplicative fragments or, equivalently, by the underlying additive and multiplicative algebras (see Figure 1). However, to date there has been no proof-theoretical analysis corresponding to this general characterisation. On the one hand, both a complete natural deduction proof system satisfying normalisation, and a complete sequent calculus satisfying cut-elimination have been given for O'Hearn and Pym's original bunched logic BI [23]. On the other hand, similarly well-behaved analogues of these syntactic proof systems for the other varieties of bunched logic have been conspicuously absent from the literature. This is less than ideal from the theoretical point of view but also from a practical perspective since, in particular, separation logic and many of the aforementioned related program analysis tools are based on BBI, the Boolean variant of BI. Usually, proof systems for BBI are formulated in a crude manner by adding a sufficiently powerful axiom or inference rule to the corresponding proof system for BI, the inclusion of which typically breaks any previously extant normalisation or cut-elimination properties. Extending the current BI proof systems to BBI (or other variants) without breaking these properties is known to be highly problematic.

A potential resolution to this technical impasse is suggested by our earlier work with Calcagno on Classical BI (CBI) [5], in which we showed that CBI could be naturally presented as a *display calculus* with the cut-elimination property. Display calculi, due to Belnap [2], are consecution calculi *à la* Gentzen which were originally employed primarily as a device for giving a disciplined proof theory to relevant and modal logics. The distinguishing feature of display calculi is the *display property*: any consecution can always be rearranged so that a given part appears alone on the appropriate side

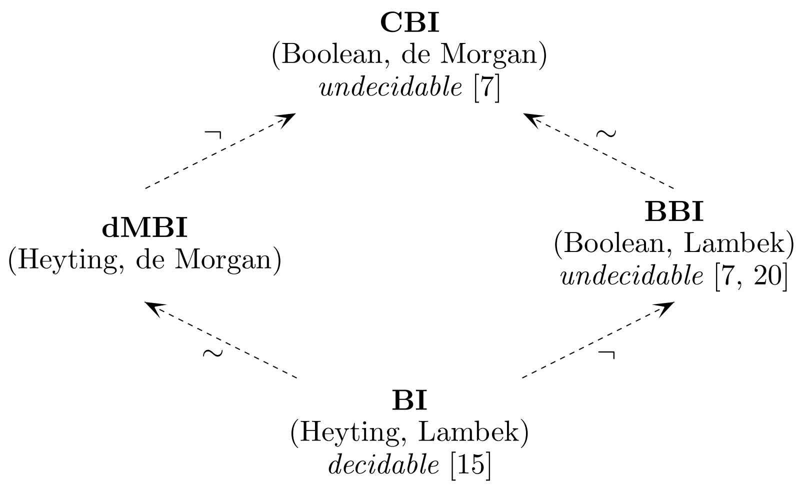

Figure 1. The bunched logic family. The (additive, multiplicative) subtitles indicate the underlying additive and multiplicative algebras. The arrows denote the addition of either additive ($\neg$) or multiplicative ($\sim$) classical negation.

of the proof turnstile. To ensure this property we need both a richer form of consecution than that of typical Gentzen-style sequents, and a set of auxiliary "display" rules for rearranging them in the required fashion. The extra complexity is compensated for by an elegant, symmetric presentation of the calculus, analogous to that of Gentzen's sequent calculi. Furthermore, Belnap showed that cut-elimination is guaranteed for any display calculus whose rules obey a set of 8 easily verifiable syntactic conditions.

In this paper, we obtain a *unified* display calculus proof theory for all four principal bunched logics in Figure 1. First, we formulate display calculi for the elementary logics which characterise the additive and multiplicative components of these four logics. Since Belnap's original display apparatus does not adapt to the intuitionistic components (because it relies on the presence of classical negation), we instead exploit the residual relationship between conjunction and implication to obtain a display property, as happens in a number of papers on display calculus [16, 27, 29]. We then obtain display calculi for the bunched logics by combining the display calculi for their additive and multiplicative components. Since these elementary calculi are entirely orthogonal to one another, combining them preserves their main desirable structural properties: the display property, cut-elimination and soundness / completeness with respect to a standard presentation of the corresponding logic. In addition to cut-elimination, we show how to constrain the use of display-equivalence in proofs so that only finitely many rearrangements of any consecution need be considered. This ensures that an exhaustive proof search in any of our calculi is finitely branching. (However, we cannot guarantee that such a proof search will terminate in general; indeed, both BBI and CBI are known to be undecidable [7, 20].) Additionally, in the case of BBI and CBI, we establish a full deduction theorem for our display calculi (analogous to that established for full propositional linear logic by Lincoln et al. [21]), showing that arbitrary theories may be faithfully encoded inside proof judgements. Finally, in the case of BI, we establish translations between cut-free proofs in our display calculus and cut-free proofs in its standard sequent calculus (given by Pym in [23]), thus demonstrating that this sequent calculus can be seen as an "optimised" version of our display calculus. The fact that our display calculi for the other bunched logics cannot be similarly optimised into sequent presentations — due to their seemingly non-eliminable use of *unary* structural connectives as well as binary ones — goes some way to explaining why, in our opinion, well-behaved sequent calculi for these logics are very unlikely to materialise.

The remainder of this paper is organised as follows. In Section 2 we present the four main bunched logics in Figure 1 as free combinations of elementary logics. Section 3 presents our unified display calculus proof theory for the principal bunched logics and their components. Section 4 covers cut-elimination for our calculi and shows how the use of display-equivalence may be constrained in proofs. Section 5 presents our deduction theorem for the BBI and CBI display calculi. In Section 6 we compare our display calculus for BI with its bunched sequent calculus. Section 7 concludes.

This is a revised and expanded journal version of a conference paper [4]. In particular, most of the material in Sections 4 and 5 is entirely new. We have endeavoured to include proofs in as much detail as space permits.

2. From elementary logics to bunched logics

In this section, we define the four principal bunched logics (cf. Figure 1) as free combinations of well-known elementary logics.

We assume a fixed infinite set $\mathcal{V}$ of propositional variables. *Formulas* are constructed from propositional variables using the logical connectives given in Figure 2: any $P \in \mathcal{V}$ is a formula, and so is the result of applying a logical connective to the appropriate number of formulas. We restrict the syntax of formulas in a given logic by stipulating which formula connectives are permitted to occur. We write F, G, H, etc., to range over formulas.

We regard a *logic* $\mathcal{L}$ as being specified by: (a) the set of logical connectives

Additive symbol	*Multiplicative symbol*	*Arity*	*Generic meaning*
$\top$	$\top^*$	0	truth
$\bot$	$\bot^*$	0	falsity
$\neg$	$\sim$	1	negation
$\wedge$	$*$	2	conjunction
$\vee$	$\overset{*}{\vee}$	2	disjunction
$\rightarrow$	$-\!\!*$	2	implication

Figure 2. Logical connectives.

$$\frac{}{F \vdash F}\ (\mathrm{Ax}) \qquad \frac{}{F \vdash \top}\ (\top) \qquad \frac{}{\bot \vdash F}\ (\bot)$$

$$\frac{}{G_i \vdash G_1 \vee G_2}\ i \in \{1,2\}\ (\vee\mathrm{I}) \qquad \frac{F \vdash G \quad F \vdash H}{F \vdash G \wedge H}\ (\wedge\mathrm{I}) \qquad \begin{array}{c} F \wedge G \vdash H \\ \hline\hline F \vdash G \rightarrow H \end{array}\ (\rightarrow)$$

$$\frac{}{G_1 \wedge G_2 \vdash G_i}\ i \in \{1,2\}\ (\wedge\mathrm{E}) \qquad \frac{F \vdash H \quad G \vdash H}{F \vee G \vdash H}\ (\vee\mathrm{E}) \qquad \frac{F \vdash G \quad G \vdash H}{F \vdash H}\ (\mathrm{MP})$$

....

$$\frac{}{\neg F \dashv\vdash F \rightarrow \bot}\ (\neg) \qquad \frac{}{\neg\neg F \vdash F}\ (\neg\neg)$$

Figure 3. Basic presentations of IL and CL. IL is by given by the axioms and rules above the dotted line. CL is obtained by adding the axioms below the dotted line.

which may occur in formulas of the logic; and (b) a basic proof system for entailments of the form $F \vdash G$, where F and G are formulas. We write an axiom with conclusion $F \dashv\vdash G$ to abbreviate two axioms with respective conclusions $F \vdash G$ and $G \vdash F$, and we write a rule with a double-line between premise and conclusion to indicate that it is *symmetric*, i.e., that the premise and conclusion may be exchanged. We specify four well-known elementary logics, which form the additive and multiplicative components of the bunched logics in Figure 1, as follows:

- Intuitionistic logic, IL, has logical connectives $\top$, $\bot$, $\wedge$, $\vee$ and $\rightarrow$. Classical logic, CL, adds the negation $\neg$. We present IL and CL in Figure 3.
- Lambek multiplicative logic, LM (a.k.a. multiplicative intuitionistic linear logic), has as logical connectives $\top^*$, $*$ and $-\!\!*$. De Morgan multiplicative logic, dMM (a.k.a. multiplicative classical linear logic), extends these by $\bot^*$, $\sim$ and $\overset{*}{\vee}$. We present LM and dMM in Figure 4.

$$\frac{}{F \vdash F}\ (\text{Ax}) \qquad \frac{}{F * (G * H) \dashv\vdash (F * G) * H}\ (\text{Assoc.})$$

$$\frac{}{F * \top^* \dashv\vdash F}\ (\top^*) \qquad \frac{}{F * G \vdash G * F}\ (\text{Comm.})$$

$$\frac{F_1 \vdash G_1 \quad F_2 \vdash G_2}{F_1 * F_2 \vdash G_1 * G_2}\ (*\text{I}) \qquad \frac{F * G \vdash H}{F \vdash G \mathrel{-\!\!*} H}\ (\mathrel{-\!\!*}) \qquad \frac{F \vdash G \quad G \vdash H}{F \vdash H}\ (\text{MP})$$

. . . .

$$\frac{}{\bot^* \dashv\vdash {\sim}\top^*}\ (\bot^*) \qquad \frac{}{F \mathbin{\overset{*}{\vee}} G \dashv\vdash {\sim}({\sim}F * {\sim}G)}\ (\overset{*}{\vee})$$

$$\frac{}{{\sim}F \dashv\vdash F \mathrel{-\!\!*} \bot^*}\ ({\sim}) \qquad \frac{}{{\sim}{\sim}F \vdash F}\ ({\sim}{\sim})$$

Figure 4. Basic presentations of LM and dMM. LM is by given by the axioms and rules above the dotted line. dMM is obtained by adding the axioms below the dotted line.

We write $\mathcal{E} = \{\text{IL}, \text{CL}, \text{LM}, \text{dMM}\}$ for this set of elementary logics. By the *free combination* $\mathcal{L}_1 + \mathcal{L}_2$ of two logics $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{E}$, we mean the logic whose logical connectives and presentation are the unions of, respectively, the logical connectives and the presentations of $\mathcal{L}_1$ and $\mathcal{L}_2$. The bunched logics $\mathcal{B} = \{\text{BI}, \text{BBI}, \text{dMBI}, \text{CBI}\}$ in Figure 1 can then be defined very straightforwardly in terms of their elementary components:

- BI, the 'logic of bunched implications' (cf. [22, 24]), is given by $\text{IL} + \text{LM}$;
- BBI, a.k.a. Boolean BI (cf. [14]), is given by $\text{CL} + \text{LM}$;
- dMBI, standing for "de Morgan BI", is given by $\text{IL} + \text{dMM}$;
- CBI, a.k.a. Classical BI (cf. [5]), is given by $\text{CL} + \text{dMM}$.

Our proof-theoretic definition of the logics in $\mathcal{E} \cup \mathcal{B}$ above will be taken as the baseline with respect to which our display calculi for these logics are later proven correct. This has the benefit of freeing our analysis from unnecessary semantic considerations. However, we note that our definitions of the bunched logics $\mathcal{B}$ can be seen to be in agreement with those found elsewhere in the literature. For example, our presentations of BI and BBI agree with their counterparts in [24] and [14] respectively. (To our knowledge, dMBI has not appeared in the literature before, while a display calculus for CBI was presented in [5]; in the next section, we will reconstruct this system as part of our unified proof theory for $\mathcal{B}$, and show it agrees with our characterisation of CBI here.)

3. Display calculi for the principal bunched logics

In this section we give display calculi for the elementary logics $\mathcal{E}$ given in Section 2, and combine them to obtain display calculi for the principal bunched logics $\mathcal{B}$. As a preliminary, we first present the basic notions that we require in order to specify a display calculus in the spirit of Belnap [2].

Structures are constructed from formulas using the structural connectives given by Figure 5: any formula is a structure, and so is the result of applying a structural connective to the appropriate number of structures. We write all unary connectives as prefix operators, and all binary connectives as infix operators. We write W, X, Y, Z, etc., to range over structures. If X and Y are structures then $X \vdash Y$ is called a *consecution*. There is a classification of the substructure occurrences in a consecution into *antecedent* and *consequent* parts, which extends the left-right division created by the proof turnstile by taking into account the "polarities" of the structural connectives.

Additive	*Multiplicative*	*Arity*	*Antecedent meaning*	*Consequent meaning*
$\emptyset$	$\varnothing$	0	truth	falsity
$\sharp$	$\flat$	1	negation	negation
;	,	2	conjunction	disjunction
$\Rightarrow$	$\multimap$	2	undefined	implication

Figure 5. Structural connectives.

Definition 3.1 (Antecedent / consequent part). Each substructure occurrence in a structure X is classified as either a *positive part* or a *negative part* of X, as follows:

- X is a positive part of X;
- if Z is a negative (positive) part of X then it is a positive (negative) part of $\sharp X$ and $\flat X$;
- if Z is a positive (negative) part of X_1 or X_2 then it is a positive (negative) part of $X_1 \,;\, X_2$ and $X_1 \,,\, X_2$;
- if Z is a negative (positive) part of X_1 or a positive (negative) part of X_2, then it is a positive (negative) part of $X_1 \Rightarrow X_2$ and $X_1 \multimap X_2$.

Z is said to be an *antecedent (consequent) part* of a consecution $X \vdash Y$ if it is a positive (negative) part of X or a negative (positive) part of Y.

Consecutions are interpreted as entailments between formulas as follows.

Definition 3.2 (Consecution validity). For any structure Z we define the formulas Ψ_Z and Υ_Z by mutual structural induction as follows:

$$\begin{array}{rclrcl}
\Psi_F & = & F & & & \\
\Psi_\emptyset & = & \top & \Psi_\varnothing & = & \top^* \\
\Psi_{\sharp X} & = & \neg\Upsilon_X & \Psi_{\flat X} & = & \sim\Upsilon_X \\
\Psi_{X;Y} & = & \Psi_X \wedge \Psi_Y & \Psi_{X,Y} & = & \Psi_X * \Psi_Y \\
\Psi_{X\Rightarrow Y} & = & \text{undefined} & \Psi_{X\multimap Y} & = & \text{undefined}
\end{array}$$

$$\begin{array}{rclrcl}
\Upsilon_F & = & F & & & \\
\Upsilon_\emptyset & = & \bot & \Upsilon_\varnothing & = & \bot^* \\
\Upsilon_{\sharp X} & = & \neg\Psi_X & \Upsilon_{\flat X} & = & \sim\Psi_X \\
\Upsilon_{X;Y} & = & \Upsilon_X \vee \Upsilon_Y & \Upsilon_{X,Y} & = & \Upsilon_X \mathbin{\overset{*}{\vee}} \Upsilon_Y \\
\Upsilon_{X\Rightarrow Y} & = & \Psi_X \to \Upsilon_Y & \Upsilon_{X\multimap Y} & = & \Psi_X \mathrel{-\!\!*} \Upsilon_Y
\end{array}$$

$X \vdash Y$ is said to be *valid* in the logic $\mathcal{L}$ iff $\Psi_X \vdash \Upsilon_Y$ is provable in $\mathcal{L}$.

We remark that, in each of our display calculi, we shall restrict the form of consecutions by stipulating which of the structural connectives may appear as the main (i.e. outermost) connective of an antecedent or consequent part. In doing so, we ensure that the restrictions on the structural connectives match the available formula connectives, so that validity of consecutions is always well defined. In particular, neither $\Rightarrow$ nor $\multimap$ will ever be permitted to appear as the main connective of an antecedent part of a consecution.

The defining feature of a display calculus is the availability of a *display-equivalence* on consecutions: an equivalence relation such that, for any antecedent (consequent) part of a given consecution, one can obtain by rearrangement an equivalent consecution in which that part appears as the *entire* antecedent (consequent).

Definition 3.3 (Display-equivalence). Let $<>_D$ be a symmetric relation on consecutions and let $\equiv_D$ be the equivalence relation given by the reflexive-transitive closure of $<>_D$. We say that $\equiv_D$ is a *display-equivalence* if, for any antecedent part Z of $X \vdash Y$, one can construct a structure W such that $X \vdash Y \equiv_D Z \vdash W$, and for any consequent part Z of $X \vdash Y$ one can construct a structure W such that $X \vdash Y \equiv_D W \vdash Z$. The process of rearranging $X \vdash Y$ into $Z \vdash W$ or $W \vdash Z$ is called *displaying* Z.

A *display calculus* $\mathrm{DL}_\mathcal{L}$ for a logic $\mathcal{L}$ is then specified by the following:

Antecedent / consequent structural connectives: The sets of structural connectives that are permitted to appear as the main connective of an antecedent / consequent part of a consecution, respectively.

Display postulates: A set of symmetric rules of the form $C <>_D C'$, where C and C' are consecutions, such that the reflexive-transitive closure $\equiv_D$ of $<>_D$ is a display-equivalence (cf. Defn. 3.3).

Logical rules: Proof rules for the formula connectives, typically divided into pairs of left- and right-introduction rules for each logical connective in the manner familiar from sequent calculus. Note that, since we can appeal to the display-equivalence $\equiv_D$, these rules may be written so that the formula introduced by a rule is displayed (alone) in its conclusion.

Structural rules: Proof rules for the structural connectives.

In addition to the logical and structural proof rules given by their specification, all our display calculi share a common set of *identity rules*:

$$\frac{}{P \vdash P}\,(\mathrm{Id}) \qquad \frac{X \vdash F \quad F \vdash Y}{X \vdash Y}\,(\mathrm{Cut}) \qquad \frac{X' \vdash Y'}{X \vdash Y}\; X \vdash Y \equiv_D X' \vdash Y' \;\; (\equiv_D)$$

where P ranges over propositional variables. We remark that a display calculus specified as above is not guaranteed to obey any particular proof-theoretic properties over and above the availability of display-equivalence; as is well-known, display calculi may fail to enjoy cut-elimination, interpolation, or decidability. However, cut-elimination is guaranteed for display calculi with sufficiently well-behaved logical and structural rules, as famously demonstrated by Belnap [2]. (In Section 4, we show that all of our display calculi meet Belnap's conditions for cut-elimination.)

We give display calculus specifications for the elementary logics IL, CL, LM and dMM in Figures 6, 7, 8 and 9 respectively. Some remarks on our formulation of these elementary display calculi are in order.

Firstly, the display postulates for the classical logics CL and dMM essentially follow Belnap [2] although, for convenience, we build in commutativity of the semicolon and comma on the left hand side of consecutions (since both $\wedge$ and $*$ are commutative). Note that, by simple manipulations, both $X \vdash Y \equiv_D X \vdash \sharp\sharp Y$ and $\sharp X \vdash Y \equiv_D X \vdash \sharp Y$ hold in $\mathrm{DL}_{\mathrm{CL}}$, and their analogues (with $\flat$ in place of $\sharp$) hold in $\mathrm{DL}_{\mathrm{dMM}}$.

Secondly, in the case of the calculi for the intuitionistic logics IL and LM, we cannot employ "classical" display postulates of the type used in $\mathrm{DL}_{\mathrm{CL}}$ and $\mathrm{DL}_{\mathrm{dMM}}$, because the required de Morgan relationships between connectives do not hold; in particular, IL and LM lack the classical negations necessary to interpret $\sharp$ and $\flat$. Instead, we allow the structural connectives $\Rightarrow$ and $\multimap$ to occur in consequent position only (interpreted as $\rightarrow$ and $-\!\!*$ respectively), and we allow semicolon and comma to occur in antecedent positions only

Antecedent structure connectives: $\emptyset \quad ;$
Consequent structure connectives: $\Rightarrow$

Display postulates: $X\,;Y \vdash Z \;<>_D\; X \vdash Y \Rightarrow Z \;<>_D\; Y\,;X \vdash Z$

Logical rules:

$$\frac{}{\bot \vdash X}\,(\bot\mathrm{L})$$

$$\frac{\emptyset \vdash X}{\top \vdash X}\,(\top\mathrm{L}) \qquad \frac{}{X \vdash \top}\,(\top\mathrm{R}) \qquad \frac{F \vdash X \quad G \vdash X}{F \vee G \vdash X}\,(\vee\mathrm{L}) \qquad \frac{X \vdash F_i}{X \vdash F_1 \vee F_2}\,\begin{matrix} i \in \{1,2\} \\ (\vee\mathrm{R}) \end{matrix}$$

$$\frac{F\,;G \vdash X}{F \wedge G \vdash X}\,(\wedge\mathrm{L}) \quad \frac{X \vdash F \quad X \vdash G}{X \vdash F \wedge G}\,(\wedge\mathrm{R}) \quad \frac{X \vdash F \quad G \vdash Y}{F \to G \vdash X \Rightarrow Y}\,(\to\mathrm{L}) \quad \frac{X\,;F \vdash G}{X \vdash F \to G}\,(\to\mathrm{R})$$

Structural rules:

$$\frac{\emptyset\,;X \vdash Y}{X \vdash Y}\,(\emptyset\mathrm{L}) \quad \frac{W\,;(X\,;Y) \vdash Z}{(W\,;X)\,;Y \vdash Z}\,(\mathrm{AAL}) \quad \frac{X \vdash Z}{X\,;Y \vdash Z}\,(\mathrm{WkL}) \quad \frac{X\,;X \vdash Y}{X \vdash Y}\,(\mathrm{CtrL})$$

Figure 6. Specification of $\mathrm{DL}_{\mathrm{IL}}$.

(interpreted as $\wedge$ and $*$). Using such consecutions, the logical rules have simple formulations, and our display postulates just capture the residual connection between implication and conjunction. The same idea is employed in a number of other works on display calculus [16, 27, 29]. Indeed, in these works, conjunction and implication are Gentzen duals, so that the *same* structural connective is used to represent conjunction in antecedent position and implication in consequent position. We use distinguished structural connectives $\Rightarrow$ and $\multimap$ for the implications to avoid confusion with the use of comma and semicolon to represent disjunction in consequent positions in CL and dMM. This approach also allows us to translate $\mathrm{DL}_{\mathcal{L}}$ consecutions into formulas using a single translation function for all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$ in Defn. 3.2.

Thirdly, note that because we allow $\emptyset$ and the semicolon to occur only in antecedent positions in $\mathrm{DL}_{\mathrm{IL}}$ consecutions, we are forced to employ structure-free formulations of the rules ($\vee$L) and ($\bot$L). For the sake of symmetry and convenience, we also use structure-free formulations of ($\wedge$R) and ($\top$R), and use the same formulations of these rules in $\mathrm{DL}_{\mathrm{CL}}$. We have chosen the structural rules of $\mathrm{DL}_{\mathrm{CL}}$ and $\mathrm{DL}_{\mathrm{dMM}}$ to be convenient, rather than minimal: for example, the rules (MAL) and (MAR) are interderivable in $\mathrm{DL}_{\mathrm{dMM}}$.

Now we obtain display calculi for $\mathcal{B}$ by defining, for $\mathcal{L}_1 \in \{\mathrm{IL}, \mathrm{CL}\}$ and $\mathcal{L}_2 \in \{\mathrm{LM}, \mathrm{dMM}\}$:

$$\mathrm{DL}_{\mathcal{L}_1+\mathcal{L}_2} =_{\mathrm{def}} \mathrm{DL}_{\mathcal{L}_1} + \mathrm{DL}_{\mathcal{L}_2}$$

where $\mathrm{DL}_{\mathcal{L}_1} + \mathrm{DL}_{\mathcal{L}_2}$ is the display calculus whose antecedent and consequent

Antecedent structure connectives: $\emptyset \quad \sharp \quad ;$
Consequent structure connectives: $\emptyset \quad \sharp \quad ;$

Display postulates:

$$\begin{array}{ccccc} X ; Y \vdash Z & <>_D & X \vdash \sharp Y ; Z & <>_D & Y ; X \vdash Z \\ X \vdash Y ; Z & <>_D & X ; \sharp Y \vdash Z & <>_D & X \vdash Z ; Y \\ X \vdash Y & <>_D & \sharp Y \vdash \sharp X & <>_D & \sharp\sharp X \vdash Y \end{array}$$

Logical rules:

$$\frac{}{\bot \vdash X}\,(\bot\text{L}) \qquad \frac{X \vdash \emptyset}{X \vdash \bot}\,(\bot\text{R}) \qquad \frac{\emptyset \vdash X}{\top \vdash X}\,(\top\text{L}) \qquad \frac{}{X \vdash \top}\,(\top\text{R})$$

$$\frac{\sharp F \vdash X}{\neg F \vdash X}\,(\neg\text{L}) \qquad \frac{X \vdash \sharp F}{X \vdash \neg F}\,(\neg\text{R}) \qquad \frac{F \vdash X \quad G \vdash X}{F \vee G \vdash X}\,(\vee\text{L}) \qquad \frac{X \vdash F ; G}{X \vdash F \vee G}\,(\vee\text{R})$$

$$\frac{F ; G \vdash X}{F \wedge G \vdash X}\,(\wedge\text{L}) \qquad \frac{X \vdash F \quad X \vdash G}{X \vdash F \wedge G}\,(\wedge\text{R}) \qquad \frac{X \vdash F \quad G \vdash Y}{F \to G \vdash \sharp X ; Y}\,(\to\text{L}) \qquad \frac{X ; F \vdash G}{X \vdash F \to G}\,(\to\text{R})$$

Structural rules:

$$\begin{array}{c} \emptyset ; X \vdash Y \\ \hline\hline X \vdash Y \end{array}\,(\emptyset\text{L}) \qquad \begin{array}{c} W ; (X ; Y) \vdash Z \\ \hline\hline (W ; X) ; Y \vdash Z \end{array}\,(\text{AAL}) \qquad \frac{X \vdash Z}{X ; Y \vdash Z}\,(\text{WkL}) \qquad \frac{X ; X \vdash Y}{X \vdash Y}\,(\text{CtrL})$$

$$\begin{array}{c} X \vdash Y ; \emptyset \\ \hline\hline X \vdash Y \end{array}\,(\emptyset\text{R}) \qquad \begin{array}{c} W \vdash (X ; Y) ; Z \\ \hline\hline W \vdash X ; (Y ; Z) \end{array}\,(\text{AAR}) \qquad \frac{X \vdash Z}{X \vdash Y ; Z}\,(\text{WkR}) \qquad \frac{X \vdash Y ; Y}{X \vdash Y}\,(\text{CtrR})$$

Figure 7. Specification of DL_{CL}.

structure connectives, display postulates, and logical and structural rules are, respectively, given by the unions of those of $\text{DL}_{\mathcal{L}_1}$ with those of $\text{DL}_{\mathcal{L}_2}$. We remark that DL_{CBI} as presented here is equivalent to its earlier formulation in [5], while DL_{BI}, DL_{BBI} and DL_{dMBI} are new. However, DL_{dMBI} is very nearly equivalent to Restall's display calculus for the well-known relevant logic **RW** obtained from **R** by removing the multiplicative contraction rule [27]. The two calculi differ only because **RW** lacks the additive intuitionistic $\to$ and $\bot$ of dMBI (which can however be added conservatively). In Section 6, we compare DL_{BI} with the BI sequent calculus.

We now demonstrate that each of our specifications does indeed give rise to a true display calculus, in the sense that the display property holds.

Proposition 3.4 (Display). *For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, the equivalence $\equiv_D$ induced by the display postulates of* $\text{DL}_{\mathcal{L}}$ *is a display-equivalence for* $\text{DL}_{\mathcal{L}}$.

Proof. We must show that an arbitrary substructure occurrence Z in a consecution $X \vdash Y$ of $\text{DL}_{\mathcal{L}}$ can be displayed (as the entire antecedent or

Antecedent structure connectives: $\varnothing \quad ,$
Consequent structure connectives: $\multimap$

Display postulates: $X, Y \vdash Z \;<>_D\; X \vdash Y \multimap Z \;<>_D\; Y, X \vdash Z$

Logical rules:

$$\frac{\varnothing \vdash X}{\top^* \vdash X}\,(\top^*\mathrm{L}) \qquad \frac{F \;,\; G \vdash X}{F * G \vdash X}\,(*\mathrm{L}) \qquad \frac{X \vdash F \quad G \vdash Y}{F \mathrel{-\!\!*} G \vdash X \multimap Y}\,(\mathrel{-\!\!*}\mathrm{L})$$

$$\frac{}{\varnothing \vdash \top^*}\,(\top^*\mathrm{R}) \qquad \frac{X \vdash F \quad Y \vdash G}{X \;,\; Y \vdash F * G}\,(*\mathrm{R}) \qquad \frac{X \;,\; F \vdash G}{X \vdash F \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}\mathrm{R})$$

Structural rules:

$$\frac{\varnothing \;,\; X \vdash Y}{X \vdash Y}\,(\varnothing\mathrm{L}) \qquad \frac{W \;,\; (X \;,\; Y) \vdash Z}{(W \;,\; X) \;,\; Y \vdash Z}\,(\mathrm{MAL})$$

Figure 8. Specification of $\mathrm{DL}_{\mathrm{LM}}$.

consequent as appropriate) using the display postulates of $\mathrm{DL}_{\mathcal{L}}$. By induction on the depth at which Z occurs in X or Y (defined in the obvious way), it suffices to show that each of the *immediate* substructures of X and Y can be displayed. This fact may be verified essentially by eye for each $\mathcal{L} \in \mathcal{E}$ (in fact, we need only check e.g. $\mathrm{DL}_{\mathrm{IL}}$ and $\mathrm{DL}_{\mathrm{CL}}$ since the consecution syntax and display postulates of $\mathrm{DL}_{\mathrm{LM}}$ and $\mathrm{DL}_{\mathrm{IL}}$ are isomorphic, as are those of $\mathrm{DL}_{\mathrm{dMM}}$ and $\mathrm{DL}_{\mathrm{CL}}$). It follows immediately that the immediate substructures of $X \vdash Y$ can be displayed for each $\mathcal{L} \in \mathcal{B}$, using the display postulates from the display calculus for the appropriate component logic. ■

THEOREM 3.5 (Soundness). *For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, if a consecution of $\mathrm{DL}_{\mathcal{L}}$ is provable in $\mathrm{DL}_{\mathcal{L}}$ then it is valid.*

PROOF. As usual, we prove that each rule of $\mathrm{DL}_{\mathcal{L}}$ preserves validity from premises to conclusion. In practice this means deriving the rule in $\mathcal{L}$ under the translation $(X \vdash Y) \mapsto (\Psi_X \vdash \Upsilon_Y)$ from consecutions to formula entailments given by Defn. 3.2. In the case of the display rule $(\equiv_D)$, it suffices to show that each individual display postulate is $\mathcal{L}$-derivable under translation. This is a long and tedious verification for each of the elementary logics $\mathcal{L} \in \mathcal{E}$. For $\mathcal{L} \in \mathcal{B}$, the local soundness property is then immediate since, clearly, if any $\mathrm{DL}_{\mathcal{L}_i}$ rule is $\mathcal{L}_i$-derivable under translation for $i \in \{1, 2\}$, then the rules of $\mathrm{DL}_{\mathcal{L}_1} + \mathrm{DL}_{\mathcal{L}_2}$ are derivable under translation in $\mathcal{L}_1 + \mathcal{L}_2$.

We show the cases of a display postulate and a logical rule, both taken from $\mathrm{DL}_{\mathrm{dMM}}$. In the following we treat $*$ as being associative and commutative, and omit explicit applications of the corresponding axioms. We also

Antecedent structure connectives: $\varnothing \quad \flat \quad ,$
Consequent structure connectives: $\varnothing \quad \flat \quad ,$

Display postulates:

$$
\begin{array}{ccccc}
X , Y \vdash Z & <>_D & X \vdash \flat Y , Z & <>_D & Y , X \vdash Z \\
X \vdash Y , Z & <>_D & X , \flat Y \vdash Z & <>_D & X \vdash Z , Y \\
X \vdash Y & <>_D & \flat Y \vdash \flat X & <>_D & \flat\flat X \vdash Y
\end{array}
$$

Logical rules:

$$
\frac{}{\bot^* \vdash \varnothing}\,(\bot^*\mathrm{L}) \qquad
\frac{X \vdash \varnothing}{X \vdash \bot^*}\,(\bot^*\mathrm{R}) \qquad
\frac{\varnothing \vdash X}{\top^* \vdash X}\,(\top^*\mathrm{L}) \qquad
\frac{}{\varnothing \vdash \top^*}\,(\top^*\mathrm{R})
$$

$$
\frac{\flat F \vdash X}{\sim F \vdash X}\,(\sim\mathrm{L}) \qquad
\frac{X \vdash \flat F}{X \vdash \sim F}\,(\sim\mathrm{R}) \qquad
\frac{F \vdash X \quad G \vdash Y}{F \mathbin{\overset{*}{\vee}} G \vdash X , Y}\,(\overset{*}{\vee}\mathrm{L}) \qquad
\frac{X \vdash F , G}{X \vdash F \mathbin{\overset{*}{\vee}} G}\,(\overset{*}{\vee}\mathrm{R})
$$

$$
\frac{F , G \vdash X}{F * G \vdash X}\,(*\mathrm{L}) \qquad
\frac{X \vdash F \quad Y \vdash G}{X , Y \vdash F * G}\,(*\mathrm{R}) \qquad
\frac{X \vdash F \quad G \vdash Y}{F \mathrel{-\!\!*} G \vdash \flat X , Y}\,(\mathrel{-\!\!*}\mathrm{L}) \qquad
\frac{X , F \vdash G}{X \vdash F \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}\mathrm{R})
$$

Structural rules:

$$
\frac{\varnothing , X \vdash Y}{\overline{X \vdash Y}}\,(\varnothing\mathrm{L}) \qquad
\frac{W , (X , Y) \vdash Z}{\overline{(W , X) , Y \vdash Z}}\,(\mathrm{MAL}) \qquad
\frac{X \vdash Y , \varnothing}{\overline{X \vdash Y}}\,(\varnothing\mathrm{R}) \qquad
\frac{W \vdash (X , Y) , Z}{\overline{W \vdash X , (Y , Z)}}\,(\mathrm{MAR})
$$

Figure 9. Specification of $\mathrm{DL}_{\mathrm{dMM}}$.

write the rule label (MP), (A) to denote an abbreviated application of (MP) of which one (suppressed) premise is an instance of the axiom (A).

Case $X \vdash Y <>_D \flat Y \vdash \flat X$. Using the definition of Ψ_- and Υ_-, it suffices to show that $F \vdash G$ and $\sim G \vdash \sim F$ are interderivable in dMM for any formulas F and G. We can derive $\sim G \vdash \sim F$ from $F \vdash G$ as follows:

$$
\dfrac{\dfrac{\dfrac{F \vdash G \qquad \dfrac{\dfrac{}{G \mathrel{-\!\!*} \bot^* \vdash G \mathrel{-\!\!*} \bot^*}\,(\mathrm{Ax})}{G \vdash (G \mathrel{-\!\!*} \bot^*) \mathrel{-\!\!*} \bot^*}\,(\mathrel{-\!\!*}) \times 2}{F \vdash (G \mathrel{-\!\!*} \bot^*) \mathrel{-\!\!*} \bot^*}\,(\mathrm{MP})}{\dfrac{G \mathrel{-\!\!*} \bot^* \vdash F \mathrel{-\!\!*} \bot^*}{G \mathrel{-\!\!*} \bot^* \vdash \sim F}\,(\mathrm{MP}), (\sim)}\,(\mathrel{-\!\!*}) \times 2}{\sim G \vdash \sim F}\,(\mathrm{MP}), (\sim)
$$

For the reverse direction, we can derive $\sim\sim F \vdash \sim\sim G$ from $\sim G \vdash \sim F$ using the first part, from which we can derive $F \vdash G$ using (MP) with the fact that $F \vdash \sim\sim F$ is (easily) derivable and $\sim\sim G \vdash G$ is an axiom.

Case ($\mathrel{-\!\!*}$L). It suffices to show that $F \mathrel{-\!\!*} G \vdash \sim A \mathbin{\overset{*}{\vee}} B$ is derivable from $A \vdash F$ and $G \vdash B$ in dMM. First, we derive $A \mathrel{-\!\!*} B \vdash \sim A \mathbin{\overset{*}{\vee}} B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \mathrel{-\!\!*} B \vdash A \mathrel{-\!\!*} B}\,(\mathrm{Ax})}{A * (A \mathrel{-\!\!*} B) \vdash B}\,(\mathrel{-\!\!*}) \qquad \dfrac{\dfrac{}{\sim B \vdash B \mathrel{-\!\!*} \bot^*}\,(\sim)}{B \vdash \sim B \mathrel{-\!\!*} \bot^*}\,(\mathrel{-\!\!*}) \times 2}{A * (A \mathrel{-\!\!*} B) \vdash \sim B \mathrel{-\!\!*} \bot^*}\,(\mathrm{MP})}{A \vdash (A \mathrel{-\!\!*} B) \mathrel{-\!\!*} (\sim B \mathrel{-\!\!*} \bot^*)}\,(\mathrel{-\!\!*})}{\sim\sim A \vdash (A \mathrel{-\!\!*} B) \mathrel{-\!\!*} (\sim B \mathrel{-\!\!*} \bot^*)}\,(\mathrm{MP}), (\sim\sim)}{A \mathrel{-\!\!*} B \vdash (\sim\sim A * \sim B) \mathrel{-\!\!*} \bot^*}\,(\mathrel{-\!\!*}) \times 3}{A \mathrel{-\!\!*} B \vdash \sim(\sim\sim A * \sim B)}\,(\mathrm{MP}), (\sim)}{A \mathrel{-\!\!*} B \vdash \sim A \mathbin{\overset{*}{\vee}} B}\,(\mathrm{MP}), (\overset{*}{\vee})$$

We can then construct the required derivation as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash F \qquad \dfrac{\dfrac{}{F \mathrel{-\!\!*} G \vdash F \mathrel{-\!\!*} G}\,(\mathrm{Ax})}{F \vdash (F \mathrel{-\!\!*} G) \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}) \times 2}{A \vdash (F \mathrel{-\!\!*} G) \mathrel{-\!\!*} G}\,(\mathrm{MP})}{(F \mathrel{-\!\!*} G) * A \vdash G}\,(\mathrel{-\!\!*}) \qquad G \vdash B}{(F \mathrel{-\!\!*} G) * A \vdash B}\,(\mathrm{MP})}{F \mathrel{-\!\!*} G \vdash A \mathrel{-\!\!*} B}\,(\mathrel{-\!\!*}) \qquad \begin{matrix}\text{(see above)} \\ \vdots \\ A \mathrel{-\!\!*} B \vdash \sim A \mathbin{\overset{*}{\vee}} B\end{matrix}}{F \mathrel{-\!\!*} G \vdash \sim A \mathbin{\overset{*}{\vee}} B}\,(\mathrm{MP})$$

This completes the case, and the proof. ■

PROPOSITION 3.6. *Any* $\mathrm{DL_{IL}}$ *derivation may be transformed into a* $\mathrm{DL_{CL}}$ *derivation by replacing all structures of the form* $X \Rightarrow Y$ *by the structure* $\sharp X; Y$ *and possibly inserting some applications of (WkR). Similarly, any* $\mathrm{DL_{LM}}$ *derivation may be transformed into a* $\mathrm{DL_{dMM}}$ *derivation by replacing all structures of the form* $X \multimap Y$ *by the structure* $\flat X, Y$.

PROOF. In the case of $\mathrm{DL_{IL}}$ and $\mathrm{DL_{CL}}$, one just verifies by eye that every proof rule and display postulate of $\mathrm{DL_{IL}}$ becomes a proof rule or display postulate of $\mathrm{DL_{CL}}$ under the uniform replacement of structures of the form $X \Rightarrow Y$ by $\sharp X; Y$. The exception to this is the $\mathrm{DL_{IL}}$ rule ($\vee$R), which is however derivable from its $\mathrm{DL_{CL}}$ analogue by applying (WkR). The case of $\mathrm{DL_{LM}}$ and $\mathrm{DL_{dMM}}$ is similar. ■

LEMMA 3.7 (Identity). *For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, and for any formula F of $\mathcal{L}$, the consecution $F \vdash F$ is provable in* $\mathrm{DL}_{\mathcal{L}}$.

PROOF. By structural induction on F, distinguishing a case for every possible logical connective of $\mathcal{L}$. In the case $F = P \in \mathcal{V}$ we are immediately done by (Id). The other cases are straightforward by induction hypothesis. ■

LEMMA 3.8. *For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, the following rules are derivable in* $\mathrm{DL}_{\mathcal{L}}$, *where Ψ_{-} and Υ_{-} are the functions given in Definition 3.2:*

$$\dfrac{X \vdash Y}{\Psi_X \vdash Y}\,(\Psi L) \qquad \dfrac{}{X \vdash \Psi_X}\,(\Psi R) \qquad \dfrac{}{\Upsilon_X \vdash X}\,(\Upsilon L) \qquad \dfrac{Y \vdash X}{Y \vdash \Upsilon_X}\,(\Upsilon R)$$

PROOF. We show the derivability of all four rules simultaneously by induction on the structure of X, distinguishing a case for each possible structural connective of X. In the case where X is a formula, we appeal to Lemma 3.7 for (ΨR) and (ΥL). The other cases are straightforward using the logical and display rules for the appropriate logic and the induction hypotheses. ■

Because of the rules (ΨR) and (ΥL), Lemma 3.8 subsumes Lemma 3.7.

THEOREM 3.9 (Completeness). *For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, if a consecution of* $\mathrm{DL}_{\mathcal{L}}$ *is valid then it is provable in* $\mathrm{DL}_{\mathcal{L}}$.

PROOF. Suppose that $X \vdash Y$ is $\mathrm{DL}_{\mathcal{L}}$-valid, i.e. that $\Psi_X \vdash \Upsilon_Y$ is $\mathcal{L}$-provable. To show that $X \vdash Y$ is $\mathrm{DL}_{\mathcal{L}}$-provable, it suffices by (Cut) and the rules (ΨR) and (ΥL) given by Lemma 3.8 to show that $\Psi_X \vdash \Upsilon_Y$ is $\mathrm{DL}_{\mathcal{L}}$-provable. In practice this simply entails showing that each proof rule of $\mathcal{L}$ is $\mathrm{DL}_{\mathcal{L}}$-derivable, which is an easy exercise for each $\mathcal{L} \in \mathcal{E}$. The result then follows immediately for all $\mathcal{L} \in \mathcal{B}$ because it is clear that, if $\mathrm{DL}_{\mathcal{L}_1}$ and $\mathrm{DL}_{\mathcal{L}_2}$ can derive every rule of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then $\mathrm{DL}_{\mathcal{L}_1} + \mathrm{DL}_{\mathcal{L}_2}$ can derive every rule of $\mathcal{L}_1 + \mathcal{L}_2$. We show a typical case, the axiom ($\stackrel{*}{\vee}$) of dMM, the two directions of which are derived in $\mathrm{DL}_{\mathrm{dMM}}$ as follows (note that we use the derived rules of Lemma 3.8):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F \vdash F}\,(\Psi\mathrm{R}) \quad \dfrac{}{G \vdash G}\,(\Psi\mathrm{R})}{F \stackrel{*}{\vee} G \vdash F, G}\,(\stackrel{*}{\vee}\mathrm{L})}{\flat F, \flat G \vdash \flat F \stackrel{*}{\vee} G}\,(\equiv_D)}{\sim F * \sim G \vdash \flat F \stackrel{*}{\vee} G}\,(\Psi\mathrm{L})}{F \stackrel{*}{\vee} G \vdash \flat(\sim F * \sim G)}\,(\equiv_D)}{F \stackrel{*}{\vee} G \vdash \sim(\sim F * \sim G)}\,(\sim\mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\flat F \vdash \sim F}\,(\Psi\mathrm{R}) \quad \dfrac{}{\flat G \vdash \sim G}\,(\Psi\mathrm{R})}{\flat F, \flat G \vdash \sim F * \sim G}\,(*\mathrm{R})}{\flat(\sim F * \sim G) \vdash F, G}\,(\equiv_D)}{\flat(\sim F * \sim G) \vdash F \stackrel{*}{\vee} G}\,(\stackrel{*}{\vee}\mathrm{R})}{\sim(\sim F * \sim G) \vdash F \stackrel{*}{\vee} G}\,(\sim\mathrm{L})$$

This completes the case, and the proof. ■

4. Cut-elimination and proof search

In this section we address the problem of constraining proof search in our display calculi $\mathrm{DL}_{\mathcal{L}}$ presented in the previous section. First, we prove that cut is eliminable in each of our calculi by demonstrating that they meet Belnap's well-known cut-elimination conditions C1–C8 (see [2]). Then we show that, in addition, applications of the display rule ($\equiv_D$) can be restricted so that only finitely many rearrangements of any consecution need be considered. As a result, all infinite branching points can be eliminated from the proof search space.

A display calculus proof is *cut-free* if it contains no instances of (Cut).

THEOREM 4.1 (Cut-elimination). *For all $\mathcal{L} \in \mathcal{E} \cup \mathcal{B}$, any $\mathrm{DL}_{\mathcal{L}}$ proof of $X \vdash Y$ can be transformed into a cut-free proof of $X \vdash Y$.*

PROOF. Given that $\mathrm{DL}_{\mathcal{L}}$ satisfies the display property (Proposition 3.4), it suffices to verify that the proof rules of $\mathrm{DL}_{\mathcal{L}}$ meet Belnap's conditions C1–C8 guaranteeing cut-elimination [2]. Since these conditions appear in many places in the literature (e.g. [2, 18]) and C1–C7 are easily verified properties of the individual proof rules, we just consider the most complicated condition C8 here. Note that a formula occurrence in an instance I of a proof rule R is said to be *principal* in I if (a) it is displayed (alone) in the conclusion of I and (b) it is not part of a structure assigned to a structure variable in our statement of the rule R.

C8: *Eliminability of matching principal formulas.* If there are inferences I_1 and I_2 with respective conclusions $X \vdash F$ and $F \vdash Y$ and with F principal in both I_1 and I_2, then either $X \vdash Y$ is one of $X \vdash F$ and $F \vdash Y$, or there is a derivation of $X \vdash Y$ from the premises of I_1 and I_2 in which every instance of cut has a cut-formula which is a proper subformula of F.

Verification. If F is a propositional variable P then $X \vdash F$ and $F \vdash Y$ are both instances of (Id). Thus we must have $(X \vdash F) = (F \vdash Y) = (X \vdash Y)$, and are done. Otherwise, by inspection of the proof rules, F is introduced in I_1 and I_2 respectively by the right and left introduction rule for the main connective of F. We show a typical case, $F = F_1 \wedge F_2$, in which case $\mathcal{L}$ contains either IL or CL and the considered principal cut is of the form:

$$\dfrac{\dfrac{X \vdash F \qquad X \vdash G}{X \vdash F \wedge G}\,(\wedge\mathrm{R}) \qquad \dfrac{F \,;\, G \vdash Y}{F \wedge G \vdash Y}\,(\wedge\mathrm{L})}{X \vdash Y}\,(\mathrm{Cut})$$

If $\mathcal{L}$ contains IL, we can reduce this cut as follows:

$$
\dfrac{X \vdash F \qquad \dfrac{\dfrac{X \vdash G \qquad \dfrac{F \,;\, G \vdash Y}{G \vdash F \Rightarrow Y}\,(\equiv_D)}{X \vdash F \Rightarrow Y}\,(\text{Cut})}{F \vdash X \Rightarrow Y}\,(\equiv_D)}{\dfrac{\dfrac{X \vdash X \Rightarrow Y}{X \,;\, X \vdash Y}\,(\equiv_D)}{X \vdash Y}\,(\text{CtrL})}\,(\text{Cut})
$$

The subcase where $\mathcal{L}$ contains CL rather than IL follows by Proposition 3.6. The cases for the other connectives are similar. This completes the verification, and the proof. ∎

From a proof search perspective, Theorem 4.1 eliminates the infinite branching points provided by the cut rule. However, the display rule introduces another source of (potentially) infinite branching in proof search. Namely, in any of DL_{CL}, DL_{dMM}, DL_{BBI}, DL_{dMBI} and DL_{CBI}, an unbounded number of $\sharp$s and/or $\flat$s can be introduced into any consecution via the display postulates $X \vdash Y <>_D \bullet Y \vdash \bullet X <>_D \bullet\bullet X \vdash Y$, where $\bullet$ is $\sharp$ or $\flat$. As a result, for any consecution of these calculi there are infinitely many other consecutions that are display-equivalent to it. (This problem does not arise for DL_{IL}, DL_{LM}, and DL_{BI}, because these calculi do not use $\sharp$ or $\flat$.)

We shall show that an exhaustive proof search need only consider display-rearrangements of consecutions which are bounded in the number of $\sharp$s and $\flat$s they contain. It follows that such a proof search is finitely branching. Our technique for reducing proofs is adapted from the one used for the same purpose by Kracht [18] and later Restall [27]. However, we require a slightly more elaborate approach in order to deal with the presence in DL_{CBI} of *both* $\sharp$ and $\flat$, which can be arbitrarily nested within structures.

First, we require an auxiliary structural rule capturing the fact that, in CBI, the additive and multiplicative negations commute (cf. [6]).

Lemma 4.2. *The following proof rule is cut-free derivable in* DL_{CBI}*:*

$$
\dfrac{\sharp\flat X \vdash Y}{\flat\sharp X \vdash Y}\;(\sharp\flat)
$$

Proof. We show how to derive one direction of the rule; the reverse direction is similar.

$$
\begin{array}{cl}
\sharp\flat X \vdash Y & \\ \hline
\flat\sharp X \vdash \flat\sharp\flat\sharp Y & (\equiv_D) \\ \hline
\flat\sharp X ; \sharp Y \vdash \flat\sharp\flat\sharp Y & (\mathrm{WkL}) \\ \hline
\sharp Y \vdash \flat\sharp\flat(\flat\sharp X ; \sharp Y) & (\equiv_D) \\ \hline
\flat\sharp X ; \sharp Y \vdash \flat\sharp\flat(\flat\sharp X ; \sharp Y) & (\mathrm{WkL}) \\ \hline
\sharp\flat(\flat\sharp X ; \sharp Y) \vdash \flat(\flat\sharp X ; \sharp Y) & (\equiv_D) \\ \hline
\flat\emptyset ; \sharp\flat(\flat\sharp X ; \sharp Y) \vdash \flat(\flat\sharp X ; \sharp Y) & (\mathrm{WkL}) \\ \hline
\flat\emptyset \vdash \flat(\flat\sharp X ; \sharp Y) ; \flat(\flat\sharp X ; \sharp Y) & (\equiv_D) \\ \hline
\flat\emptyset \vdash \flat(\flat\sharp X ; \sharp Y) & (\mathrm{CtrR}) \\ \hline
\flat\sharp X \vdash Y ; \emptyset & (\equiv_D) \\ \hline
\flat\sharp X \vdash Y & (\emptyset\mathrm{R})
\end{array}
$$

■

DEFINITION 4.3. For any structure X, define the structures $\overline{\sharp X}$ and $\overline{\flat X}$ by:

$$
\overline{\sharp X} = \begin{cases} Z & \text{if } X = \sharp Z \\ \flat Z & \text{if } X = \flat\sharp Z \\ \sharp X & \text{otherwise} \end{cases}
\qquad
\overline{\flat X} = \begin{cases} Z & \text{if } X = \flat Z \\ \sharp Z & \text{if } X = \sharp\flat Z \\ \flat X & \text{otherwise} \end{cases}
$$

DEFINITION 4.4. For each $\mathcal{L} \in \{\mathrm{CL}, \mathrm{dMM}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$, we write $\mathrm{DL}^{+}_{\mathcal{L}}$ for the display calculus obtained from $\mathrm{DL}_{\mathcal{L}}$ as follows:

- if $\mathcal{L} \in \{\mathrm{CL}, \mathrm{BBI}, \mathrm{CBI}\}$, replacing the rule ($\rightarrow$L) with its alternate version ($\rightarrow$L'), and if $\mathcal{L} \in \{\mathrm{dMM}, \mathrm{dMBI}, \mathrm{CBI}\}$, replacing the rule ($-\!\!*$L) with its alternate version ($-\!\!*$L'), where ($\rightarrow$L') and ($-\!\!*$L') are as follows:

$$
\frac{X \vdash F \quad G \vdash Y}{F \rightarrow G \vdash \overline{\sharp X} ; Y}\ (\rightarrow\mathrm{L'}) \qquad \frac{X \vdash F \quad G \vdash Y}{F -\!\!* G \vdash \overline{\flat X} , Y}\ (-\!\!*\mathrm{L'})
$$

- in the case $\mathcal{L} = \mathrm{CBI}$ only, adding the extra display postulate:

$$
\sharp\flat X \vdash Y <>_D \flat\sharp X \vdash Y
$$

LEMMA 4.5. *For $\mathcal{L} \in \{\mathrm{CL}, \mathrm{BBI}, \mathrm{CBI}\}$, each of ($\rightarrow$L) and ($\rightarrow$L') is cut-free derivable from the other in $\mathrm{DL}_{\mathcal{L}}$. Similarly, for $\mathcal{L} \in \{\mathrm{dMM}, \mathrm{dMBI}, \mathrm{CBI}\}$, each of ($-\!\!*$L) and ($-\!\!*$L') is cut-free derivable from the other in $\mathrm{DL}_{\mathcal{L}}$.*

PROOF. Here we just show the interderivability of ($-\!\!*$L) and ($-\!\!*$L'); the case of ($\rightarrow$L) and ($\rightarrow$L') is similar. There are three cases to consider.

Case X not of the form $\flat Z$ or $\sharp\flat Z$. In this case we have $\overline{\flat X} = \flat X$, so the rules ($-\!\!*$L') and ($-\!\!*$L) are identical and we are trivially done.

Case $X = \flat Z$. We have $\overline{\flat X} = Z$. Each of ($-\!\!*$L) and ($-\!\!*$L') is easily cut-free derivable from the other in $\mathrm{DL_{dMM}}$ by observing that $(F -\!\!* G \vdash \flat\flat Z, Y) \equiv_D (F -\!\!* G \vdash Z, Y)$ and using the display rule. The cases $\mathcal{L}$ = dMBI and $\mathcal{L}$ = CBI follow immediately.

Case $X = \sharp\flat Z$. We have $\mathcal{L}$ = CBI and $\overline{\flat X} = \sharp Z$. Then, using the derived $\mathrm{DL_{CBI}}$ rule of Lemma 4.2, each of ($-\!\!*$L) and ($-\!\!*$L') is cut-free derivable from the other as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\sharp\flat Z \vdash F \quad G \vdash Y}{F -\!\!* G \vdash \flat\sharp\flat Z, Y}\,(-\!\!*\mathrm{L})}{\sharp\flat Z \vdash \flat F -\!\!* G, Y}\,(\equiv_D)}{\flat\sharp Z \vdash \flat F -\!\!* G, Y}\,(\sharp\flat)}{F -\!\!* G \vdash \sharp Z, Y}\,(\equiv_D) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\sharp\flat Z \vdash F \quad G \vdash Y}{F -\!\!* G \vdash \sharp Z, Y}\,(-\!\!*\mathrm{L'})}{\flat\sharp Z \vdash \flat F -\!\!* G, Y}\,(\equiv_D)}{\sharp\flat Z \vdash \flat F -\!\!* G, Y}\,(\sharp\flat)}{F -\!\!* G \vdash \flat\sharp\flat Z, Y}\,(\equiv_D)$$

This completes the proof. ∎

We remark that, as a consequence of Lemmas 4.2 and 4.5, cut-free provability in $\mathrm{DL}_\mathcal{L}$ and in $\mathrm{DL}^+_\mathcal{L}$ coincide for all $\mathcal{L} \in \{\mathrm{CL}, \mathrm{dMM}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$.

Definition 4.6 ($\sharp\flat$-reduction). A structure is said to be *$\sharp\flat$-reduced* if it does not contain any substructures of the form $\sharp\sharp X$, $\flat\flat X$, $\sharp\flat\sharp X$ or $\flat\sharp\flat X$. The *$\sharp\flat$-reduction* $r(Z)$ of a structure Z is the (unique) $\sharp\flat$-reduced structure obtained by iteratively replacing, outermost-first, all substructure occurrences in Z of the form $\sharp X$ and $\flat X$ by the structures $\overline{\sharp X}$ and $\overline{\flat X}$ respectively until the result is $\sharp\flat$-reduced.

A consecution $X \vdash Y$ is said to be $\sharp\flat$-reduced if both X and Y are $\sharp\flat$-reduced, and we define $r(X \vdash Y) =_{\mathrm{def}} r(X) \vdash r(Y)$. A $\mathrm{DL}_\mathcal{L}$ proof is said to be $\sharp\flat$-reduced if every consecution occurring in it is $\sharp\flat$-reduced.

Lemma 4.7. *For all $\mathcal{L} \in \{\mathrm{CL}, \mathrm{dMM}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$, it holds in $\mathrm{DL}^+_\mathcal{L}$ that $C \equiv_D r(C)$ for all consecutions C.*

Proof. We just show that the reduction $r(-)$ can be mimicked using the display postulates. This is easy to see for $\mathcal{L} \in \{\mathrm{CL}, \mathrm{dMM}, \mathrm{BBI}, \mathrm{dMBI}\}$: where $r(-)$ reduces a structure of the form $\sharp\sharp Z$ or $\flat\flat Z$ to Z, we may do the same by using the display postulates to eliminate the "$\sharp\sharp$" or "$\flat\flat$" in the obvious way. When $\mathcal{L}$ = CBI, we have the additional complication that

$r(-)$ may also reduce $\flat\sharp\flat Z$ to $\sharp Z$ or $\sharp\flat\sharp Z$ to $\flat Z$. In such cases, we first use the extra display postulate $\sharp\flat X \vdash Y <>_D \flat\sharp X \vdash Y$ of $\mathrm{DL}^+_{\mathrm{CBI}}$ to commute $\sharp$ and $\flat$ as needed, then eliminate the "$\sharp\sharp$" or "$\flat\flat$" as before. ∎

We are now in a position to prove our main result concerning proof search in the refined versions of our display calculi.

THEOREM 4.8. *For all* $\mathcal{L} \in \{\mathrm{CL}, \mathrm{dMM}, \mathrm{BBI}, \mathrm{dMBI}, \mathrm{CBI}\}$*, a consecution* C *has a* $\mathrm{DL}_{\mathcal{L}}$ *proof if and only if* $r(C)$ *has a cut-free,* $\sharp\flat$*-reduced* $\mathrm{DL}^+_{\mathcal{L}}$ *proof.*

PROOF. ($\Leftarrow$) Given a $\mathrm{DL}^+_{\mathcal{L}}$ proof of $r(C)$, we easily have a $\mathrm{DL}^+_{\mathcal{L}}$ proof of C by inserting an application of ($\equiv_D$), since $C \equiv_D r(C)$ in $\mathrm{DL}^+_{\mathcal{L}}$ by Lemma 4.7. This may be transformed into a $\mathrm{DL}_{\mathcal{L}}$ proof of C by using Lemma 4.5 to replace any uses of ($\rightarrow$L') and ($-\!*$L') by $\mathrm{DL}_{\mathcal{L}}$ derivations involving their counterparts ($\rightarrow$L) and ($-\!*$L), and using Lemma 4.2 to replace uses of the extra $\mathrm{DL}^+_{\mathrm{CBI}}$ display postulate $\sharp\flat X \vdash Y <>_D \flat\sharp X \vdash Y$ by $\mathrm{DL}_{\mathrm{CBI}}$ derivations.

($\Rightarrow$) Let π be a $\mathrm{DL}_{\mathcal{L}}$ proof of C. By cut-elimination (Theorem 4.1) we may assume without loss of generality that π is cut-free. By Lemma 4.5, π may be converted to a cut-free $\mathrm{DL}^+_{\mathcal{L}}$ proof π' of C by replacing all instances of ($\rightarrow$L) and ($-\!*$L) by cut-free $\mathrm{DL}^+_{\mathcal{L}}$ derivations involving their counterparts ($\rightarrow$L') and ($-\!*$L').

Now let π'' be the result of replacing every consecution appearing in π' by its $\sharp\flat$-reduction. Clearly π'' is cut-free and $\sharp\flat$-reduced by construction, and has $r(C)$ as its root. It just remains to check that π'' is still a $\mathrm{DL}^+_{\mathcal{L}}$ proof. To see this, we just observe that every proof rule instance of $\mathrm{DL}^+_{\mathcal{L}}$ remains an instance of the same rule under $\sharp\flat$-reduction of its premises and conclusion. This is obvious for the rules that do not introduce new occurrences of $\sharp$ or $\flat$ into their conclusion, which only leaves the modified implication rules ($\rightarrow$L') and ($-\!*$L') and the display rule ($\equiv_D$). The rules ($\rightarrow$L') and ($-\!*$L') are all right because the structures $\overline{\sharp X}$ and $\overline{\flat X}$ introduced into their respective conclusions are $\sharp\flat$-reduced if X is already $\sharp\flat$-reduced. Finally, for any instance of the display rule ($\equiv_D$) with premise C_1 and conclusion C_2 we have that $C_1 \equiv_D C_2$ in $\mathrm{DL}_{\mathcal{L}}$. It obviously holds that $C_1 \equiv_D C_2$ in $\mathrm{DL}^+_{\mathcal{L}}$ as well, so by Lemma 4.7 we have $r(C_1) \equiv_D C_1 \equiv_D C_2 \equiv_D r(C_2)$ in $\mathrm{DL}^+_{\mathcal{L}}$ as required. This completes the proof. ∎

Cut-free proofs in our display calculi are easily seen to enjoy the usual subformula property (in fact, this is Belnap's condition C1). Thus, as is clear by inspection of the proof rules, for any consecution C there are only finitely many $\sharp\flat$-reduced consecutions that can be obtained as premises of a proof rule instance with conclusion C. Thus an exhaustive backwards search for a $\sharp\flat$-reduced proof of a ($\sharp\flat$-reduced) consecution is finitely branching.

However, due to the structural rules, the structural analogue of the subformula property does not hold even for cut-free, $\sharp\flat$-reduced proofs: the premises of a rule instance may contain structures which are not substructures of any structure in the conclusion. Thus, like in linear logic, cut-elimination for our display calculi does not necessarily entail decidability or interpolation[1]. Indeed, both BBI and CBI have recently been shown to be undecidable [7, 20]. On the other hand, for *particular* display calculi it is possible to ensure that an exhaustive proof search is indeed terminating (cf. [27]). In particular, this should be possible for $\mathrm{DL}_{\mathrm{BI}}$, very likely using techniques similar to those employed by Restall [27], since BI is known to be decidable [15]. We believe that $\mathrm{DL}_{\mathrm{dMBI}}$ is likely decidable too. However, Kracht showed that it is sadly impossible to decide whether an arbitrary display calculus is decidable [18].

5. Deduction theorem for $\mathrm{DL}_{\mathrm{BBI}}$ and $\mathrm{DL}_{\mathrm{CBI}}$

In this section, we prove a classical deduction theorem for $\mathrm{DL}_{\mathrm{BBI}}$ and $\mathrm{DL}_{\mathrm{CBI}}$, akin to the one for propositional linear logic in [21]. That is, we show that when arbitrary theories are added as new axioms to these systems, their expressive power does not increase.

In the following, when we are required to produce derivations in both $\mathrm{DL}_{\mathrm{BBI}}$ and $\mathrm{DL}_{\mathrm{CBI}}$, we just present derivations in $\mathrm{DL}_{\mathrm{BBI}}$, since these can be transformed into suitable $\mathrm{DL}_{\mathrm{CBI}}$ derivations using Proposition 3.6. For the sake of readability, we sometimes use multiple rule labels to denote an abbreviated sequence of rule applications, and we implicitly treat the semicolon and comma as being associative and commutative, rather than explicitly applying the appropriate rules. We also frequently use the derived rules given by Lemma 3.8.

LEMMA 5.1. *The following consecutions are derivable in* $\mathrm{DL}_{\mathrm{BBI}}$ *and* $\mathrm{DL}_{\mathrm{CBI}}$ *for any formulas* F, G, H*:*

1. $\varnothing; F \vdash (\top^* \wedge F) * (\top^* \wedge F)$
2. $(\varnothing; F), (G; H) \vdash ((\top^* \wedge F) * G) \wedge H$

[1] Of course this is a general phenomenon of consecution calculi, and the same could be said, e.g., of any sequent calculus with a contraction rule. We tend to regard this as a strength rather than a deficiency of such calculi, since cut-elimination is obviously still desirable even for undecidable logics.

3. $((\varnothing; F), G); H \vdash (\top^* \wedge F) * (G \wedge H)$

Proof. We first show that $(\varnothing; F), (\varnothing; \sharp F) \vdash Z$ is $\mathrm{DL_{BBI}}$-derivable for any formula F and structure Z.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F \vdash F}\,(\Psi\mathrm{R})}{F \vdash F; Z}\,(\mathrm{WkR})}{\sharp F; F \vdash Z}\,(\equiv_D)}{(\varnothing, \sharp F); (F, \varnothing) \vdash Z}\,(\equiv_D), (\varnothing\mathrm{L})}{((\varnothing; F), (\varnothing; \sharp F)); ((\varnothing; F), (\varnothing; \sharp F)) \vdash Z}\,(\equiv_D), (\mathrm{WkL})}{(\varnothing; F), (\varnothing; \sharp F) \vdash Z}\,(\mathrm{CtrL})$$

We now derive the three consecutions given in the lemma.

1. In the following, we write β to abbreviate the formula $(\top^* \wedge F) * (\top^* \wedge F)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varnothing \vdash \top^*}\,(\top^*\mathrm{R})}{\varnothing; \sharp((\varnothing; F) \multimap \beta) \vdash \top^*}\,(\mathrm{WkL}) \quad \dfrac{\begin{array}{c}\text{(see derivation above)}\\ \vdots \\ (\varnothing; \sharp F), (\varnothing; F) \vdash \beta\end{array}}{\varnothing; \sharp((\varnothing; F) \multimap \beta) \vdash F}\,(\equiv_D)}{\varnothing; \sharp((\varnothing; F) \multimap \beta) \vdash \top^* \wedge F}\,(\wedge\mathrm{R}) \quad \dfrac{}{\varnothing, F \vdash \top^* \wedge F}\,(\Psi\mathrm{R})}{(\varnothing; \sharp((\varnothing; F) \multimap \beta)), (\varnothing; F) \vdash \beta}\,(*\mathrm{R})}{\varnothing \vdash ((\varnothing; F) \multimap \beta); ((\varnothing; F) \multimap \beta)}\,(\equiv_D)}{\varnothing \vdash (\varnothing; F) \multimap \beta}\,(\mathrm{CtrR})}{\varnothing, (\varnothing; F) \vdash \beta}\,(\equiv_D)}{\varnothing; F \vdash \beta}\,(\varnothing\mathrm{L})$$

2.

$$\dfrac{\dfrac{\dfrac{}{(\varnothing; F), G \vdash (\top^* \wedge F) * G}\,(\Psi\mathrm{R})}{(\varnothing; F), (G; H) \vdash (\top^* \wedge F) * G}\,(\equiv_D), (\mathrm{WkL}) \quad \dfrac{\dfrac{\dfrac{}{H \vdash H}\,(\Psi\mathrm{R})}{\varnothing, H \vdash H}\,(\varnothing\mathrm{L})}{(\varnothing; F), (G; H) \vdash H}\,(\equiv_D), (\mathrm{WkL})}{(\varnothing; F), (G; H) \vdash ((\top^* \wedge F) * G) \wedge H}\,(\wedge\mathrm{R})$$

3. In the following, we write γ to abbreviate the formula $(\top^* \wedge F) * (G \wedge H)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\text{(see derivation above)}\\ \vdots \\ (\varnothing;\sharp F),(\varnothing;F) \vdash G \multimap \gamma\end{matrix}}{(\varnothing;\sharp F),(\varnothing;F),G \vdash \gamma}\;(\equiv_D)}{(\varnothing;\sharp F),(((\varnothing;F),G);H) \vdash \gamma}\;(\equiv_D),(\text{WkL})}{\varnothing;\sharp((((\varnothing;F),G);H) \multimap \gamma) \vdash F}\;(\equiv_D) \qquad \dfrac{\dfrac{}{\varnothing \vdash \top^*}\;(\top^*\text{R})}{\varnothing;\sharp((((\varnothing;F),G);H) \multimap \gamma) \vdash \top^*}\;(\text{WkL})}{\begin{matrix}\varnothing;\sharp((((\varnothing;F),G);H) \multimap \gamma) \vdash \top^* \wedge F \\ \vdots \\ \text{(contd. below)}\end{matrix}}\;(\wedge\text{R})
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\text{(contd. above)}\\ \vdots \\ \varnothing;\sharp((((\varnothing;F),G);H) \multimap \gamma) \vdash \top^* \wedge F\end{matrix} \qquad \dfrac{\dfrac{\dfrac{}{G;H \vdash G \wedge H}\;(\Psi\text{R})}{(\varnothing,G);H \vdash G \wedge H}\;(\equiv_D),(\varnothing\text{L})}{((\varnothing;F),G);H \vdash G \wedge H}\;(\equiv_D),(\text{WkL})}{(\varnothing;\sharp((((\varnothing;F),G);H) \multimap \gamma)),(((\varnothing;F),G);H) \vdash \gamma}\;(*\text{R})}{\varnothing \vdash (((\varnothing;F),G);H) \multimap \gamma;(((\varnothing;F),G);H) \multimap \gamma}\;(\equiv_D)}{\varnothing \vdash (((\varnothing;F),G);H) \multimap \gamma}\;(\text{CtrR})}{\varnothing,(((\varnothing;F),G);H) \vdash \gamma}\;(\equiv_D)}{((\varnothing;F),G);H \vdash \gamma}\;(\varnothing\text{L})
$$

■

Corollary 5.2. *The following rules are derivable in* $\mathrm{DL_{BBI}}$ *and* $\mathrm{DL_{CBI}}$ *for any formula* F *and structures* X, Y, Z*:*

$$
\dfrac{(\varnothing;F),(\varnothing;F) \vdash Z}{(\varnothing;F) \vdash Z}\;\textit{(Ded1)} \qquad \dfrac{(\varnothing;F),(X;Y) \vdash Z}{((\varnothing;F),X);Y \vdash Z}\;\textit{(Ded2)}
$$

Proof. We show how to derive one direction of (Ded2) using part 2 of Lemma 5.1; the first rule and the other direction of this rule are derived similarly using parts 1 and 3 respectively.

$$
\dfrac{\begin{matrix}\text{(Lemma 5.1, pt. 2)}\\ \vdots \\ ((\varnothing;F),X);Y \vdash (\top^* \wedge F) * (\Psi_X \wedge \Psi_Y)\end{matrix} \qquad \dfrac{(\varnothing;F),(X;Y) \vdash Z}{(\top^* \wedge F) * (\Psi_X \wedge \Psi_Y) \vdash Z}\;(\Psi\text{L})}{((\varnothing;F),X);Y \vdash Z}\;(\text{Cut})
$$

■

If C is a $\mathrm{DL}_{\mathcal{L}}$ consecution, we write $\mathrm{DL}_{\mathcal{L}}{+}C$ for the proof system obtained by extending $\mathrm{DL}_{\mathcal{L}}$ with an axiom (0-premise) rule with conclusion C.

Theorem 5.3 (Deduction theorem). *Let $\mathcal{L} \in \{\mathrm{BBI}, \mathrm{CBI}\}$, and let $W \vdash Z$ be a $\mathrm{DL}_{\mathcal{L}}$ consecution. Then $X \vdash Y$ is provable in $\mathrm{DL}_{\mathcal{L}} + (W \vdash Z)$ if and only if $(\varnothing; \Psi_W \mathrel{-\!\!*} \Upsilon_Z), X \vdash Y$ is provable in $\mathrm{DL}_{\mathcal{L}}$.*

Proof. ($\Leftarrow$) Given a $\mathrm{DL}_{\mathcal{L}}$ proof of $(\varnothing; \Psi_W \mathrel{-\!\!*} \Upsilon_Z), X \vdash Y$, we directly construct a proof of $X \vdash Y$ in the extended system $\mathrm{DL}_{\mathcal{L}} + (W \vdash Z)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{W \vdash Z}}{W \vdash \Upsilon_Z}\,(\Upsilon\mathrm{R})}{\Psi_W \vdash \Upsilon_Z}\,(\Psi\mathrm{L})}{\varnothing, \Psi_W \vdash \Upsilon_Z}\,(\varnothing\mathrm{L})}{\varnothing \vdash \Psi_W \mathrel{-\!\!*} \Upsilon_Z}\,(\mathrel{-\!\!*}\mathrm{R}) \qquad \dfrac{\begin{matrix}\vdots \\ (\varnothing; \Psi_W \mathrel{-\!\!*} \Upsilon_Z), X \vdash Y\end{matrix}}{\Psi_W \mathrel{-\!\!*} \Upsilon_Z \vdash \sharp\varnothing; X \multimap Y}\,(\equiv_D)}{\varnothing \vdash \sharp\varnothing; X \multimap Y}\,(\mathrm{Cut})}{\varnothing; \varnothing \vdash X \multimap Y}\,(\equiv_D)}{\varnothing \vdash X \multimap Y}\,(\mathrm{CtrL})}{\varnothing, X \vdash Y}\,(\equiv_D)}{X \vdash Y}\,(\varnothing\mathrm{L})
$$

($\Rightarrow$) By assumption we have a proof of $X \vdash Y$ in $\mathrm{DL}_{\mathcal{L}} + (W \vdash Z)$. Without loss of generality, we consider applications of the display rule $(\equiv_D)$ in this proof to abbreviate sequences of applications of individual display postulates. We show by induction on the height of the proof of $X \vdash Y$ that $(\varnothing; \Psi_W \mathrel{-\!\!*} \Upsilon_Z), X \vdash Y$ is provable in $\mathrm{DL}_{\mathcal{L}}$. We distinguish a case for each proof rule and display postulate of $\mathrm{DL}_{\mathrm{BBI}}$ and $\mathrm{DL}_{\mathrm{CBI}}$, and for the new axiom $W \vdash Z$. We show some of the more interesting cases in Figure 10. ■

6. Relationship between display and sequent calculi for BI

Of the four bunched logics $\mathcal{B}$, only BI is known to possess a sequent calculus with cut-elimination, given by Pym [23]. Thus it is natural to compare this calculus, LBI, with our display calculus $\mathrm{DL}_{\mathrm{BI}}$. The sequents of LBI are of the form $\Gamma \vdash F$ where F is a BI-formula and Γ is a *bunch*, given by:

$$\Gamma ::= F \mid \emptyset \mid \varnothing \mid \Gamma \,;\, \Gamma \mid \Gamma \,,\, \Gamma$$

$$\dfrac{\dfrac{\dfrac{}{\Psi_W \mathrel{-\!\!*} \Upsilon_Z \vdash W \multimap Z}\;(\Upsilon\mathrm{L})}{\varnothing; \Psi_W \mathrel{-\!\!*} \Upsilon_Z \vdash W \multimap Z}\;(\mathrm{WkL})}{(\varnothing; \Psi_W \mathrel{-\!\!*} \Upsilon_Z)\,, W \vdash Z}\;(\equiv_D)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}(\mathrm{I.H.})\\ \vdots \\ (\varnothing;\alpha), X \vdash F\end{matrix} \qquad \dfrac{\begin{matrix}(\mathrm{I.H.})\\ \vdots \\ (\varnothing;\alpha), G \vdash Y\end{matrix}}{G \vdash (\varnothing;\alpha) \multimap Y}\;(\equiv_D)}{F \mathrel{-\!\!*} G \vdash ((\varnothing;\alpha), X) \multimap ((\varnothing;\alpha) \multimap Y)}\;(\mathrel{-\!\!*}\mathrm{L})}{(\varnothing;\alpha), (\varnothing;\alpha) \vdash (F \mathrel{-\!\!*} G) \multimap (X \multimap Y)}\;(\equiv_D)}{\dfrac{\varnothing;\alpha \vdash (F \mathrel{-\!\!*} G) \multimap (X \multimap Y)}{(\varnothing;\alpha), F \mathrel{-\!\!*} G \vdash X \multimap Y}\;(\equiv_D)}\;(\mathrm{Ded1})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}(\mathrm{I.H.})\\ \vdots \\ (\varnothing;\alpha), X \vdash F\end{matrix} \qquad \dfrac{\begin{matrix}(\mathrm{I.H.})\\ \vdots \\ (\varnothing;\alpha), G \vdash Y\end{matrix}}{G \vdash ((\varnothing;\alpha) \multimap Y)}\;(\equiv_D)}{F \to G \vdash \sharp((\varnothing;\alpha), X); ((\varnothing;\alpha) \multimap Y)}\;(\to\mathrm{L})}{((\varnothing;\alpha), X); F \to G \vdash (\varnothing;\alpha) \multimap Y}\;(\equiv_D)}{(\varnothing;\alpha), (F \to G; X) \vdash (\varnothing;\alpha) \multimap Y}\;(\mathrm{Ded2})}{((\varnothing;\alpha), F \to G); X \vdash (\varnothing;\alpha) \multimap Y}\;(\mathrm{Ded2})}{(\varnothing;\alpha), (((\varnothing;\alpha), F \to G); X) \vdash Y}\;(\equiv_D)}{((\varnothing;\alpha), (\varnothing;\alpha), F \to G); X \vdash Y}\;(\mathrm{Ded2})}{\dfrac{(\varnothing;\alpha), (\varnothing;\alpha) \vdash (F \to G) \multimap (\sharp X; Y)}{\dfrac{\varnothing;\alpha \vdash (F \to G) \multimap (\sharp X; Y)}{(\varnothing;\alpha), F \to G \vdash \sharp X; Y}\;(\equiv_D)}\;(\mathrm{Ded1})}\;(\equiv_D)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}(\mathrm{I.H.})\\ \vdots \\ (\varnothing;\alpha), \sharp Y \vdash \sharp X\end{matrix}}{\sharp Y \vdash (\varnothing;\alpha) \multimap \sharp X}\;(\equiv_D)}{X; \sharp Y \vdash (\varnothing;\alpha) \multimap \sharp X}\;(\mathrm{WkL})}{(\varnothing;\alpha), (X; \sharp Y) \vdash \sharp X}\;(\equiv_D)}{((\varnothing;\alpha), X); \sharp Y \vdash \sharp X}\;(\mathrm{Ded2})}{X \vdash Y; \sharp((\varnothing;\alpha), X)}\;(\equiv_D)}{\varnothing, X \vdash Y; \sharp((\varnothing;\alpha), X)}\;(\varnothing\mathrm{L})}{\varnothing \vdash X \multimap (Y; \sharp((\varnothing;\alpha), X))}\;(\equiv_D)}{\dfrac{\varnothing;\alpha \vdash X \multimap (Y; \sharp((\varnothing;\alpha), X))}{\dfrac{((\varnothing;\alpha), X); ((\varnothing;\alpha), X) \vdash Y}{(\varnothing;\alpha), X \vdash Y}\;(\mathrm{CtrL})}\;(\equiv_D)}\;(\mathrm{WkL})$$

Figure 10. Various cases of the proof of the ($\Rightarrow$) direction of Theorem 5.3: top left, the new axiom $W \vdash Z$; top right, ($\mathrel{-\!\!*}$L); bottom left, ($\to$L); bottom right, one direction of the display postulate $X \vdash Y <>_D \sharp Y \vdash \sharp X$. In all cases α is an abbreviation for the formula $\Psi_W \mathrel{-\!\!*} \Upsilon_Z$, and (I.H.) denotes a proof given by induction hypothesis.

where F ranges over BI-formulas. *Coherent equivalence*, $\equiv$, is defined on bunches as the least congruence closed under the equations:

$$
\begin{array}{rcl@{\qquad}rcl}
(\Gamma_1 \,;\, \Gamma_2) \,;\, \Gamma_3 & \equiv & \Gamma_1 \,;\, (\Gamma_2 \,;\, \Gamma_3) & (\Gamma_1 \,,\, \Gamma_2) \,,\, \Gamma_3 & \equiv & \Gamma_1 \,,\, (\Gamma_2 \,,\, \Gamma_3) \\
\Gamma_1 \,;\, \Gamma_2 & \equiv & \Gamma_2 \,;\, \Gamma_1 & \Gamma_1 \,,\, \Gamma_2 & \equiv & \Gamma_2 \,,\, \Gamma_1 \\
\Gamma & \equiv & \emptyset \,;\, \Gamma & \Gamma & \equiv & \varnothing \,,\, \Gamma
\end{array}
$$

The right-introduction rules for the logical connectives have standard intuitionistic formulations. The left-introduction rules, and the structural rules, are written so as to apply to formulas occurring at arbitrary positions within a bunch, using the notation $\Gamma(\Delta)$ for a bunch Γ with a distinguished sub-bunch occurrence Δ. We present the rules of LBI in Figure 11.

Note that bunches are exactly the structures that can occur as antecedent parts of $\mathrm{DL_{BI}}$ consecutions. Thus every LBI sequent is a $\mathrm{DL_{BI}}$ consecution, and the left hand side of any $\mathrm{DL_{BI}}$ consecution is a bunch. We demonstrate a correspondence between cut-free proofs in the sequent calculus LBI and in our display calculus $\mathrm{DL_{BI}}$.

LEMMA 6.1. *For any* LBI *sequent* $\Gamma \vdash F$ *there is an injective, constructive map from* LBI *proofs of* $\Gamma \vdash F$ *to* $\mathrm{DL_{BI}}$ *proofs of* $\Gamma \vdash F$. *Moreover, this map preserves cut-freeness of proofs.*

PROOF. We show that each of the proof rules of LBI is derivable in $\mathrm{DL_{BI}}$. The right-introduction rules of LBI have direct equivalents in $\mathrm{DL_{BI}}$. Applications of the rule (Equiv) for coherent equivalence are translated into $\mathrm{DL_{BI}}$ as combinations of the display-equivalence rule $(\equiv_D)$, the associativity rules (AAL) and (MAL) and the unit rules ($\emptyset$L) and ($\varnothing$L).

The left-introduction rules, and the structural rules can be seen in $\mathrm{DL_{BI}}$ as a "macro" for first displaying the active part of the conclusion, then applying the corresponding left-introduction rule of $\mathrm{DL_{BI}}$ and finally reversing the original display process to restore the bunch context. E.g., we derive the ($-\!\!*$L) rule of LBI as follows:

$$
\dfrac{\dfrac{\Delta \vdash F_1 \qquad \dfrac{\Gamma(F_2) \vdash F}{F_2 \vdash X}\,(\equiv_D)}{F_1 -\!\!* F_2 \vdash \Delta \multimap X}\,(-\!\!*\mathrm{L})}{\Gamma(\Delta \,,\, F_1 -\!\!* F_2) \vdash F}\,(\equiv_D)
$$

where X is a placeholder for the consequent structure that results from displaying Z in $\Gamma(Z) \vdash F$. The other left-rules are similar. ■

Identity rules:

$$\frac{}{F \vdash F}\,(\mathrm{Id}) \qquad \frac{\Delta \vdash G \quad \Gamma(G) \vdash F}{\Gamma(\Delta) \vdash F}\,(\mathrm{Cut})$$

Logical rules:

$$\frac{}{\Gamma(\bot) \vdash F}\,(\bot\mathrm{L}) \qquad \frac{\Gamma(\emptyset) \vdash F}{\Gamma(\top) \vdash F}\,(\top\mathrm{L}) \qquad \frac{}{\Gamma \vdash \top}\,(\top\mathrm{R})$$

$$\frac{\Gamma(F\,;G) \vdash H}{\Gamma(F \wedge G) \vdash H}\,(\wedge\mathrm{L}) \qquad \frac{\Gamma(F) \vdash H \quad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\,(\vee\mathrm{L}) \qquad \frac{\Delta \vdash F \quad \Gamma(\Delta\,;G) \vdash H}{\Gamma(\Delta\,;F \to G) \vdash H}\,(\to\mathrm{L})$$

$$\frac{\Gamma \vdash F \quad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\,(\wedge\mathrm{R}) \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\,i \in \{1,2\}(\vee\mathrm{R}) \qquad \frac{\Gamma\,;F \vdash G}{\Gamma \vdash F \to G}\,(\to\mathrm{R})$$

$$\frac{\Gamma(\varnothing) \vdash F}{\Gamma(\top^*) \vdash F}\,(\top^*\mathrm{L}) \qquad \frac{\Gamma(F\,,G) \vdash H}{\Gamma(F * G) \vdash H}\,(*\mathrm{L}) \qquad \frac{\Delta \vdash F \quad \Gamma(G) \vdash H}{\Gamma(\Delta\,,F \mathrel{-\!\!*} G) \vdash H}\,(\mathrel{-\!\!*}\mathrm{L})$$

$$\frac{}{\varnothing \vdash \top^*}\,(\top^*\mathrm{R}) \qquad \frac{\Gamma \vdash F \quad \Delta \vdash F}{\Gamma\,,\Delta \vdash F * G}\,(*\mathrm{R}) \qquad \frac{\Gamma\,,F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}\mathrm{R})$$

Structural rules:

$$\frac{\Gamma(\Delta) \vdash F}{\Gamma(\Delta;\Delta') \vdash F}\,(\mathrm{WkL}) \qquad \frac{\Gamma(\Delta;\Delta) \vdash F}{\Gamma(\Delta) \vdash F}\,(\mathrm{CtrL}) \qquad \frac{\Gamma' \vdash F}{\Gamma \vdash F}\;\Gamma \equiv \Gamma'\ (\mathrm{Equiv})$$

Figure 11. The LBI sequent calculus.

Definition 6.2. For any $\mathrm{DL_{BI}}$ consecution $X \vdash Y$ define its *display-normal form* $\ulcorner X \vdash Y \urcorner$ to be the consecution obtained by applying transformations

$$\begin{array}{rcl} X \vdash Y \Rightarrow Z & \mapsto & X\,;Y \vdash Z \\ X \vdash Y \multimap Z & \mapsto & X\,,Y \vdash Z \end{array}$$

until no further such transformations are possible. Note that for any $\mathrm{DL_{BI}}$ consecution $X \vdash Y$ we have that $X \vdash Y \equiv_D \ulcorner X \vdash Y \urcorner$ and $\ulcorner X \vdash Y \urcorner$ is a unique LBI sequent of the form $\Gamma(X) \vdash F$.

Lemma 6.3. *For any* $\mathrm{DL_{BI}}$ *consecution* $X \vdash Y$ *there is a constructive map from* $\mathrm{DL_{BI}}$ *proofs of* $X \vdash Y$ *to* LBI *proofs of* $\ulcorner X \vdash Y \urcorner$*. Moreover, this map preserves cut-freeness of proofs.*

Proof. We show that each proof rule of $\mathrm{DL_{BI}}$ is derivable in LBI under the translation $\ulcorner - \urcorner$. For example, in the case of the $\mathrm{DL_{BI}}$ rule ($-\!\!*$L) we have:

$$\frac{\ulcorner X \vdash F \urcorner \quad \ulcorner G \vdash Y \urcorner}{\ulcorner F -\!\!* G \vdash X -\!\!\circ Y \urcorner} \quad = \quad \frac{X \vdash F \quad \Gamma(G) \vdash H}{\Gamma(X \;,\; F -\!\!* G) \vdash H}$$

and we are immediately done since the translated rule instance is simply the ($-\!\!*$L) rule of LBI. The other rules are similar. In the case of the display rule we treat each display postulate individually: applications of display postulates either collapse under $\ulcorner - \urcorner$ or boil down to the commutativity of the comma or semicolon, which is handled by the LBI rule (Equiv). ■

Corollary 6.4. *Any cut-elimination procedure for* $\mathrm{DL_{BI}}$ *may be constructively transformed into a cut-elimination procedure for* LBI*, and vice versa.*

Proof. For the first direction, given a proof of $\Gamma \vdash F$ in LBI we can construct a cut-free proof of $\Gamma \vdash F$ in $\mathrm{DL_{BI}}$ using Lemma 6.1 and the assumed cut-elimination procedure for $\mathrm{DL_{BI}}$, whence by Lemma 6.3 we can construct a cut-free LBI proof of $\ulcorner \Gamma \vdash F \urcorner = \Gamma \vdash F$. The other direction is similar, using the additional fact that $X \vdash Y \equiv_D \ulcorner X \vdash Y \urcorner$ in $\mathrm{DL_{BI}}$. ■

While Lemma 6.1 demonstrates that a cut-free LBI proof is essentially a cut-free $\mathrm{DL_{BI}}$ proof with some display steps omitted, Lemma 6.3 indicates the converse: any cut-free $\mathrm{DL_{BI}}$ proof can be viewed as a cut-free LBI proof by bringing each consecution into a “display-normal form”. We suggest that analogous normal forms probably do not exist in any meaningful sense for $\mathrm{DL_{BBI}}$, $\mathrm{DL_{dMBI}}$ and $\mathrm{DL_{CBI}}$ (and so Lemma 6.3 does not adapt), because of the seemingly essential presence of the structural negations $\sharp$ and/or $\flat$ in these calculi. While a single such connective can be straightforwardly eliminated in the presence of a single associative and commutative binary connective (e.g., $X \;;\; (\sharp Y \;;\; Z) \vdash W$ can be reexpressed as $X \;;\; Z \vdash Y \;;\; W$ in our calculi using display and associativity rules), it is far from clear whether we can do without them when structures are fundamentally tree-like rather than “flat”. For example, in the setting of $\mathrm{DL_{BBI}}$, if we consider the consecution $F \;,\; \sharp G \vdash \sharp H$ then it is clear that there is no structurally equivalent consecution to this one in which $\sharp$ does not occur. Thus any cut-free sequent calculus for BBI without such a unary negative structuring must represent cut-free $\mathrm{DL_{BBI}}$ proofs in a rather non-trivial way, and it appears more than likely that attempts to formulate such a calculus are fundamentally doomed — an observation borne out by our own experience and that of others [23]. Similar remarks apply to dMBI and CBI. (Of course, this does not rule out other, less syntax-directed approaches such as labelled deduction based on tableaux [15, 19] or hybrid logics [26].)

7. Conclusion

Our main contribution in this paper is a unified proof theory for the principal varieties of bunched logic, formulated using display calculi. As far as we know, this represents the first proof-theoretic treatment of bunched logic as a whole to appear in the literature. In particular, we provide the first cut-free proof system for BBI, which underlies separation and spatial logics employed in program analysis, and we (incidentally) substantiate O'Hearn and Pym's suggestion that display logic technology might apply to BI [22]. We demonstrate cut-elimination for each of our calculi, as well as soundness and completeness with respect to basic presentations of the corresponding logics. In addition, we show that the use of display-equivalence can be controlled so that only finitely many rearrangements of any consecution need be considered during a proof, and in the case of BBI and CBI we establish a full deduction theorem for our display calculi. Finally, we establish a translation between cut-free proofs in our display calculus for BI and those in its standard bunched sequent calculus. By doing so, we observe not only that this sequent calculus can be seen as an optimised display calculus, but also that the display calculi for the other bunched logics cannot be pared down to a sequent calculus in the same way. These observations provide additional evidence that our formulation of the proof theory of bunched logics in terms of display calculi is indeed canonical.

The fact that each bunched logic can individually be presented as a display calculus is relatively unsurprising in light of the earlier display calculus for CBI presented in [5], and the intuitionistic display technology, based on residuated pairs of connectives, to be found in [16, 27, 29]. As well as realising these calculi explicitly, we obtain our proof theory in a unified and economical way, by first formulating and then combining calculi for the elementary additive and multiplicative components of the bunched logics. Our treatment takes advantage of the compositionality of the display property and of Belnap's cut-elimination conditions: given that these properties hold for two "elementary" display calculi $\mathrm{DL}_{\mathcal{L}_1}$ and $\mathrm{DL}_{\mathcal{L}_2}$, it is easy to establish that the same properties hold of their orthogonal combination $\mathrm{DL}_{\mathcal{L}_1} + \mathrm{DL}_{\mathcal{L}_2}$.

Though complete cut-free proof systems for bunched logic are of clear theoretical interest, from the practical perspective it remains to be seen whether our proof theory will find application in automated theorem-proving tools. The need for such tools is quite real, e.g., in the setting of separation logic, which is based on BBI, but since both separation logic and BBI are fundamentally undecidable [7], compromises are clearly necessary. (In fact, separation logic is significantly more complicated than pure BBI, as it must also account for specific properties of the heap-like models on which it is based.) We suggest that our work might be applied in two main directions. First, the display property intuitively corresponds to "pointing" or "focusing" in a proof attempt, where one selects part of a subgoal to work on. Thus our display calculi might well find application in semi-automated or interactive proof assistants, where the proof search is partially or wholly guided by the user. Second, it might be possible to obtain useful fully-automated but incomplete proof search tools by imposing constraints on the use of structural rules. A further possibility might be to look at obtaining *deep inference* calculi, which abandon the distinction between logical connectives and structural ones [8], for bunched logics by attempting to extract formula-rewriting rules from their cut-free display calculi. Our approach may also open new avenues for display-style proof theories for other computer science logics.

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