CMIC Journal Club

R. Woods (2003) Neuroimage
Characterising Volume and Surface Deformations in an Atlas Framework

Ged Ridgway

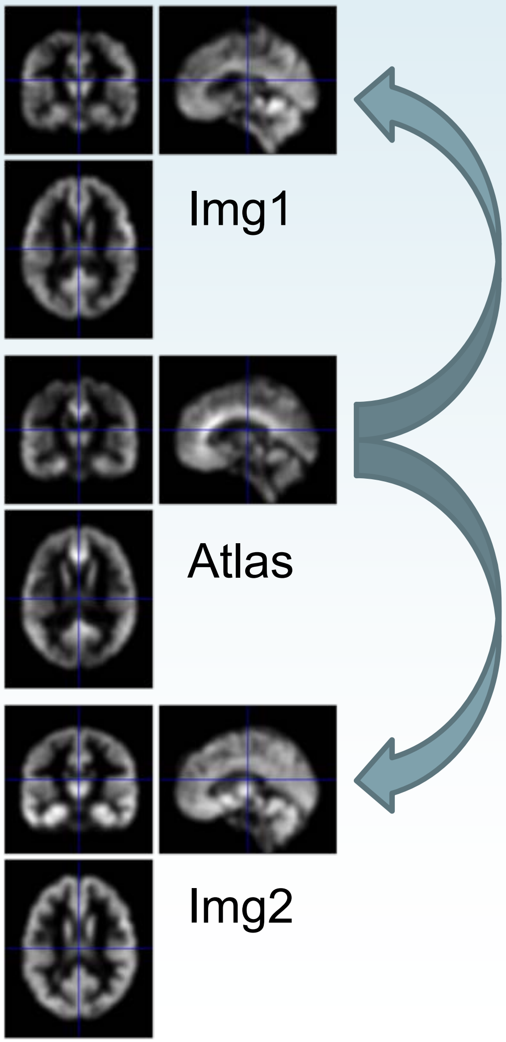
About this paper

- Very mathematical
  - Sorry (a little bit)
  - But few equations
- No results
  - But a lot of novelty
- Many useful concepts/relations to other work
  - Principal Geodesic Analysis (M-Reps, etc.)
  - Principled (diffusion) tensor analysis (Batchelor)
  - Diffeomorphisms

About this talk

- Aim: to help understand paper
  - Less focussed on paper’s content, more on ideas
  - Less critical evaluation than usual
- Some extra (simpler) maths and examples
- Some stuff glossed over
  - Surface deformations

Tensor-based morphometry



$$T_1(x,y,z) = \begin{bmatrix} T_1^x(x,y,z) \\ T_1^y(x,y,z) \\ T_1^z(x,y,z) \end{bmatrix}$$

$$J_1(x,y,z) = \begin{bmatrix} \frac{\partial T_1^x}{\partial x} & \frac{\partial T_1^x}{\partial y} & \frac{\partial T_1^x}{\partial z} \\ \frac{\partial T_1^y}{\partial x} & \frac{\partial T_1^y}{\partial y} & \frac{\partial T_1^y}{\partial z} \\ \frac{\partial T_1^z}{\partial x} & \frac{\partial T_1^z}{\partial y} & \frac{\partial T_1^z}{\partial z} \end{bmatrix}$$

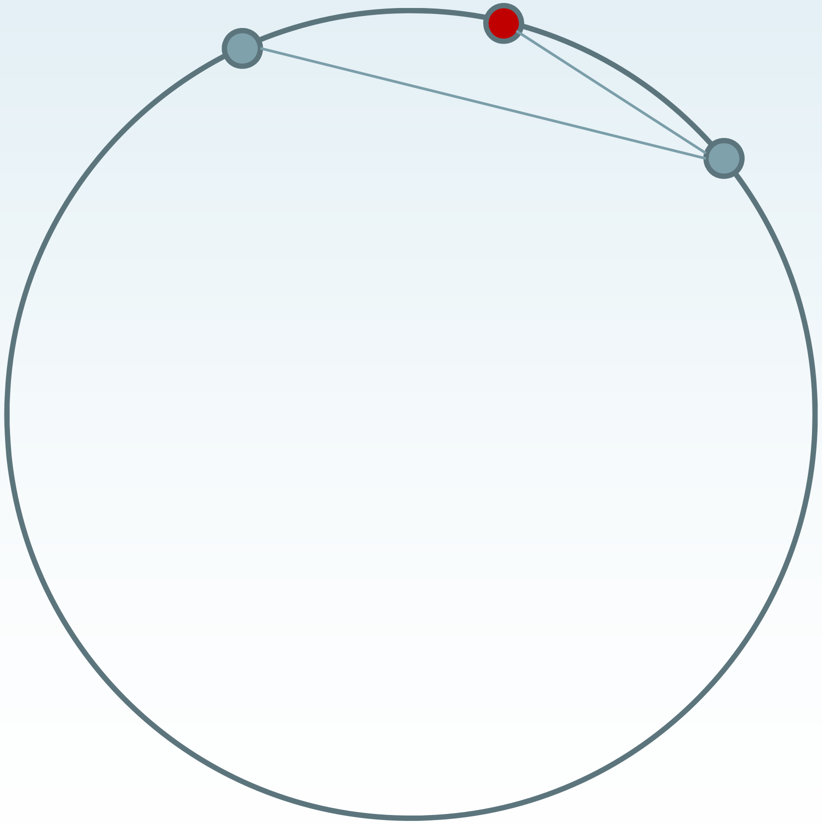
Det(J_n(x,y,z)) gives relative volume change from atlas to nth subject at (x,y,z)

Can test statistical significance of e.g. group difference at each voxel

Tensor-based morphometry

- Lots of papers on Tensor-based morphometry analyse the (scalar) determinant of the Jacobian tensor
- There is more information in the tensor than just volume change
  - E.g. two reciprocal rescalings can preserve volume
- Woods' presents a framework for analysis of the complete 3x3 Jacobian matrix
  - Accounting for the manifold in which Jacobians live...

Some simple (trick) questions

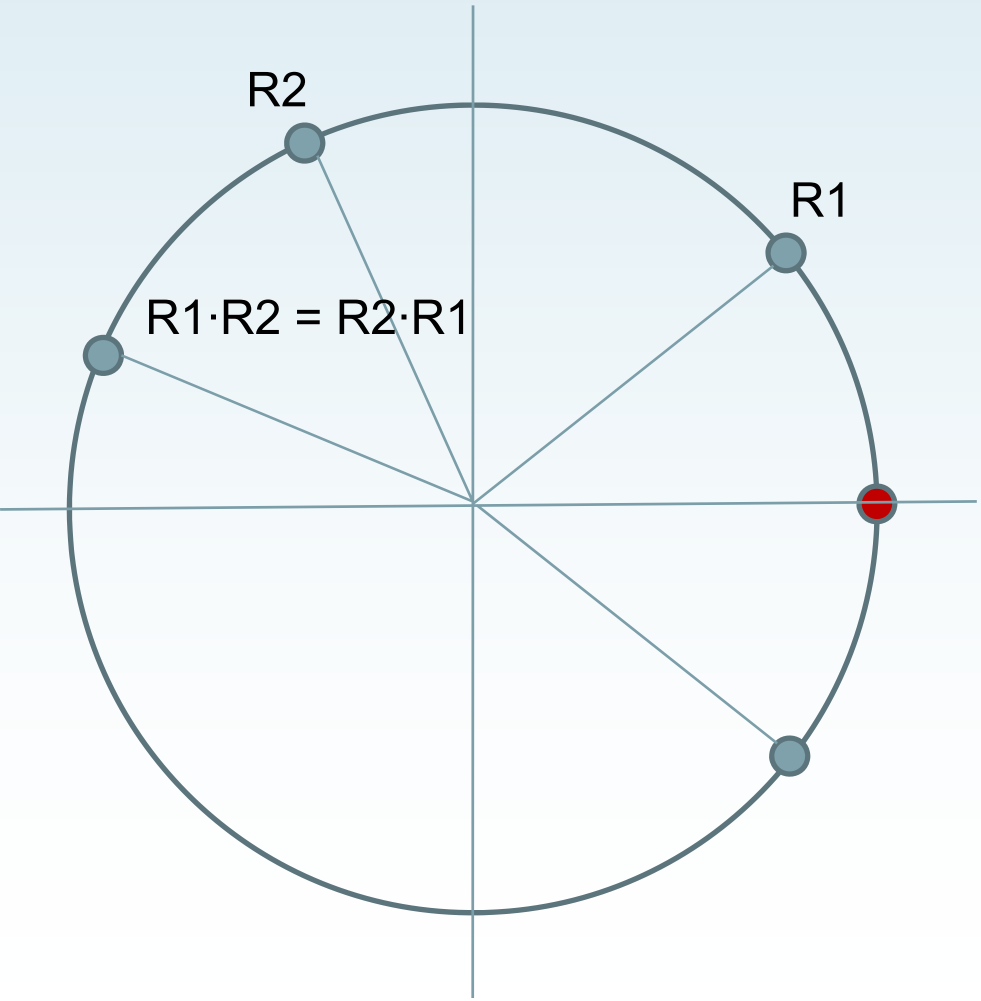


- What is the distance between these two points?
- Where is their mean?
- What about now?

Lie groups and algebras

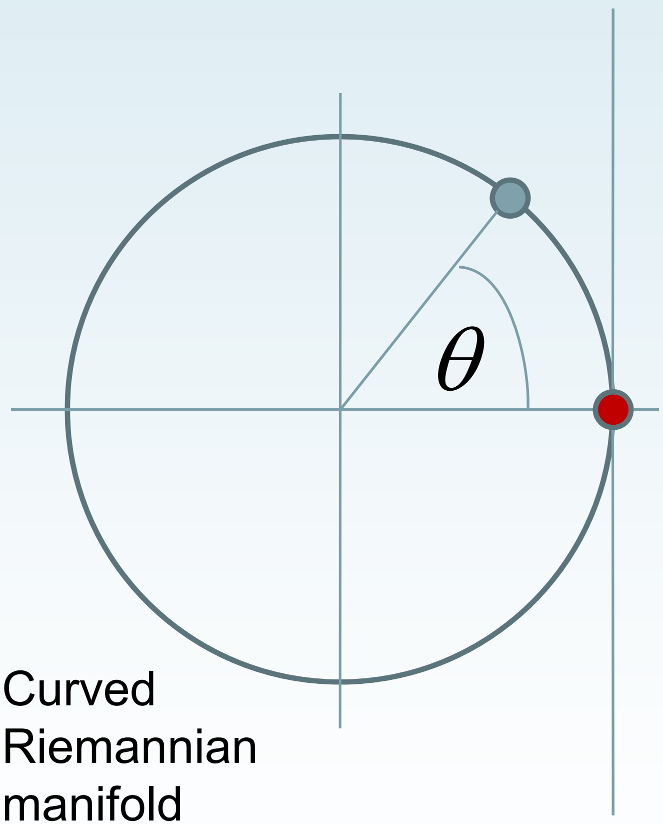
- A Lie group is a mathematical group
  - also a finite-dimensional smooth manifold
  - smooth group operations (multiplication and inversion)
- Can associate a Lie algebra
  - whose underlying vector space is the tangent space of G at the identity element
  - completely captures the local structure of the group
  - can think of elements of the Lie algebra as elements of the group that are "infinitesimally close" to the identity

The unit circle as a Lie group



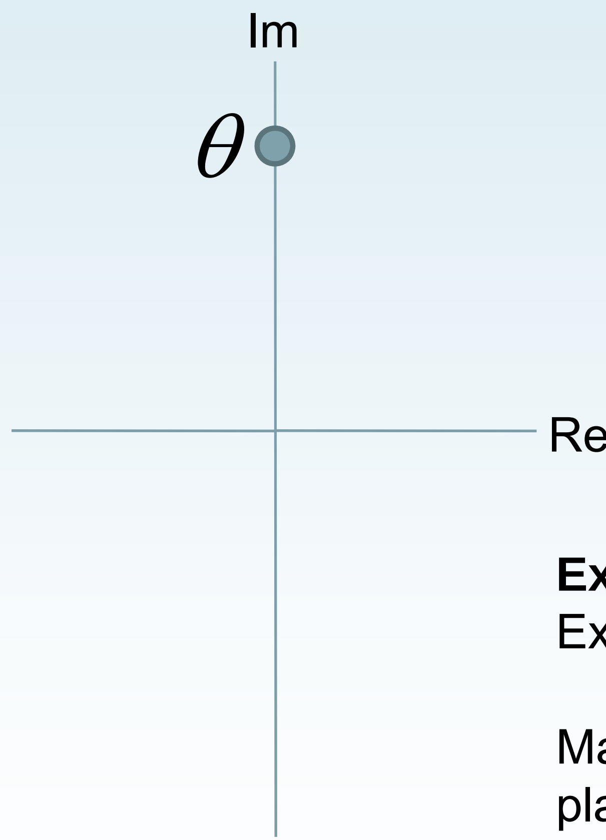
- Points on the circle are rotated versions of (1,0) or 0rad
- Composition of two elements gives another
- There is an identity 0rad
- There is an inverse -R

The unit circle's Lie algebra



Curved
Riemannian
manifold

Tangent-plane
at the identity



Tangent-plane
identified as Im

1D vector space
(flat/Euclidean)

Exponential map
$\mathrm{Exp}(\theta) = \exp(i\ \theta)$

Maps from tangent
plane to manifold
(smoothly)

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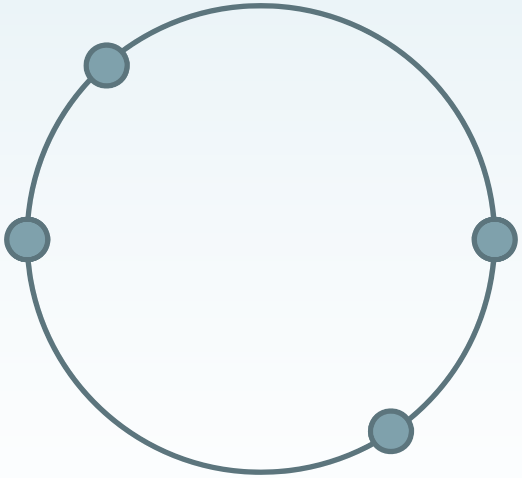
From circles to spheres

- In the example of angles around the unit circle concepts of distance and average are simple
- Mostly...
  - What is the average of 0 and pi?
  - When can we be sure of a unique mean?



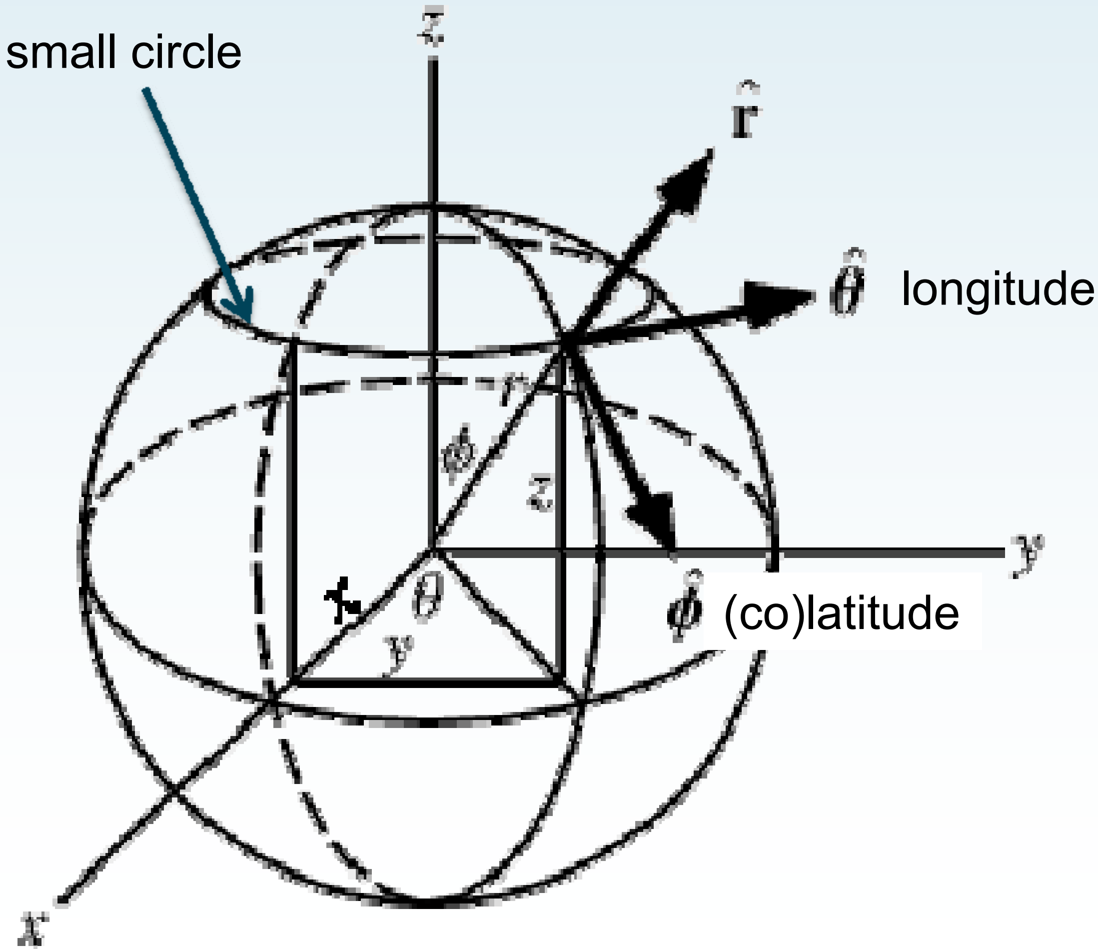
- Things aren't so simple for two angles on a sphere

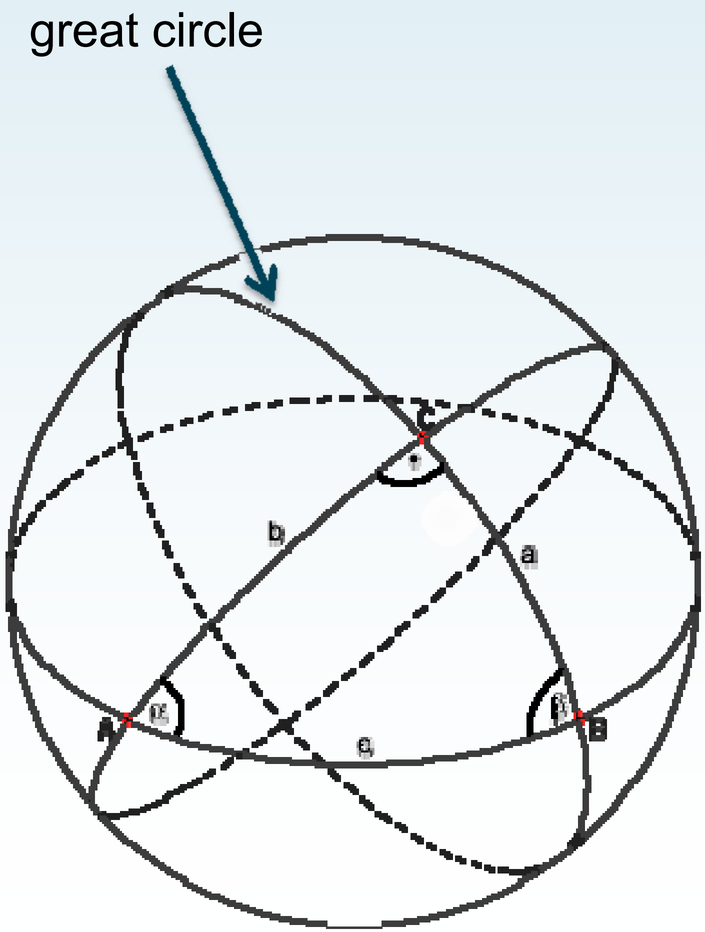
Distances on a sphere

- Consider two points
  - What is the distance between them?
  - Where is their average?

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{pmatrix} \pi/4 \\ 0 \end{pmatrix} \qquad \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{pmatrix} \pi/4 \\ \pi/2 \end{pmatrix} \qquad \begin{pmatrix} \pi/4 \\ \pi/4 \end{pmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix} \ ?$$

From circles to spheres (and back)





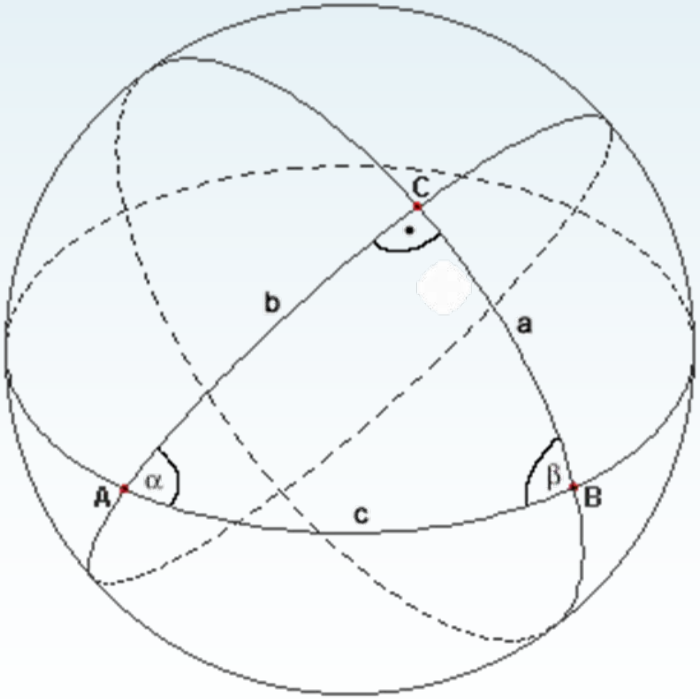
Distances on a sphere

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{pmatrix} \pi/4 \\ 0 \end{pmatrix} \qquad \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{pmatrix} \pi/4 \\ \pi/2 \end{pmatrix}$$

- Actually, the angle between these points is not pi/2
  - This would be the difference in their longitude (aka azimuth) regardless of their colatitude
  - The correct angle can be found from their scalar product
    - acos(v1'*v2) = acos(1/2) = pi/3 = 60 degrees
  - Their mean is harder to find, but the vector product gives the axis of rotation which describes the geodesic (great circle)

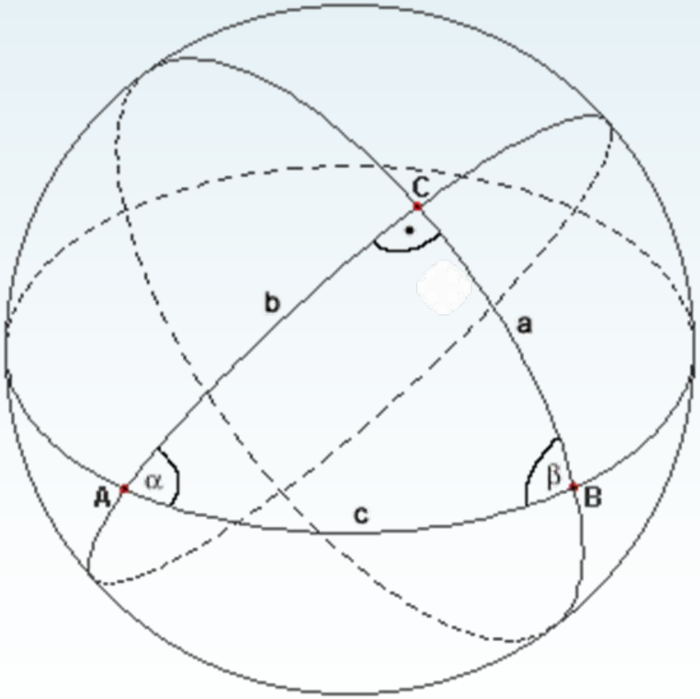
Manifolds and geodesics

- A sphere is a Riemannian manifold
- Distances need to be measured in the surface
- Geodesics are the shortest paths
- The “Fréchet mean” minimises the sum of squared geodesic dists



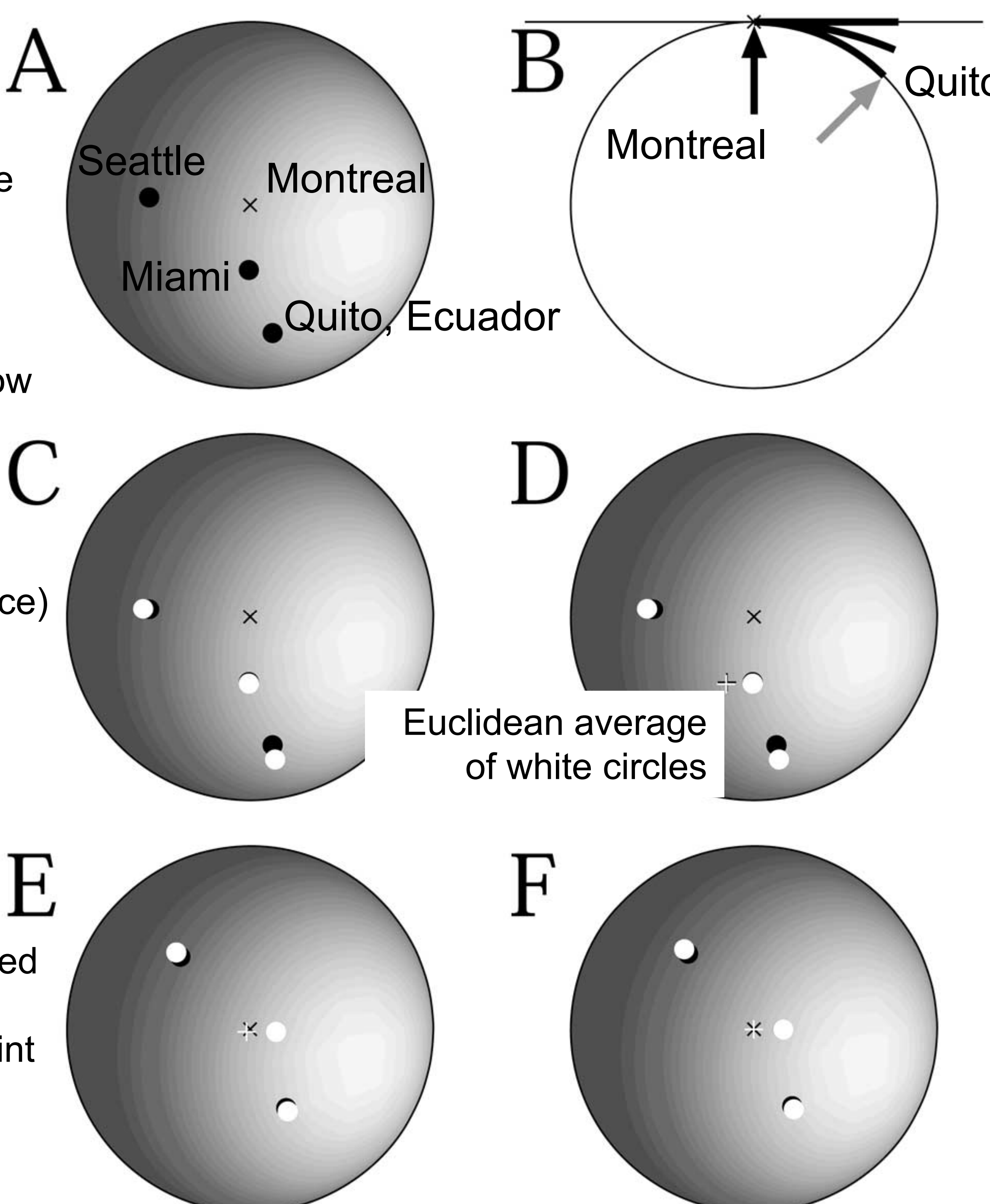
Manifolds and geodesic means

- In a plane, the sum of displacement vectors from the mean to each point is zero
- For a Riemannian manifold, the sum of velocity vectors in the tangent plane is zero
  - Defines a Karcher mean



The sphere has been rotated so Montreal (X) is closest to you the tangent-plane at X is parallel to the page.

Cross-section of A along the great circle passing through X and Quito (furthest city) Illustrates how the Geodesic between these points can be “lifted” from the manifold of the sphere to a unique point on tangent-plane (using Log)

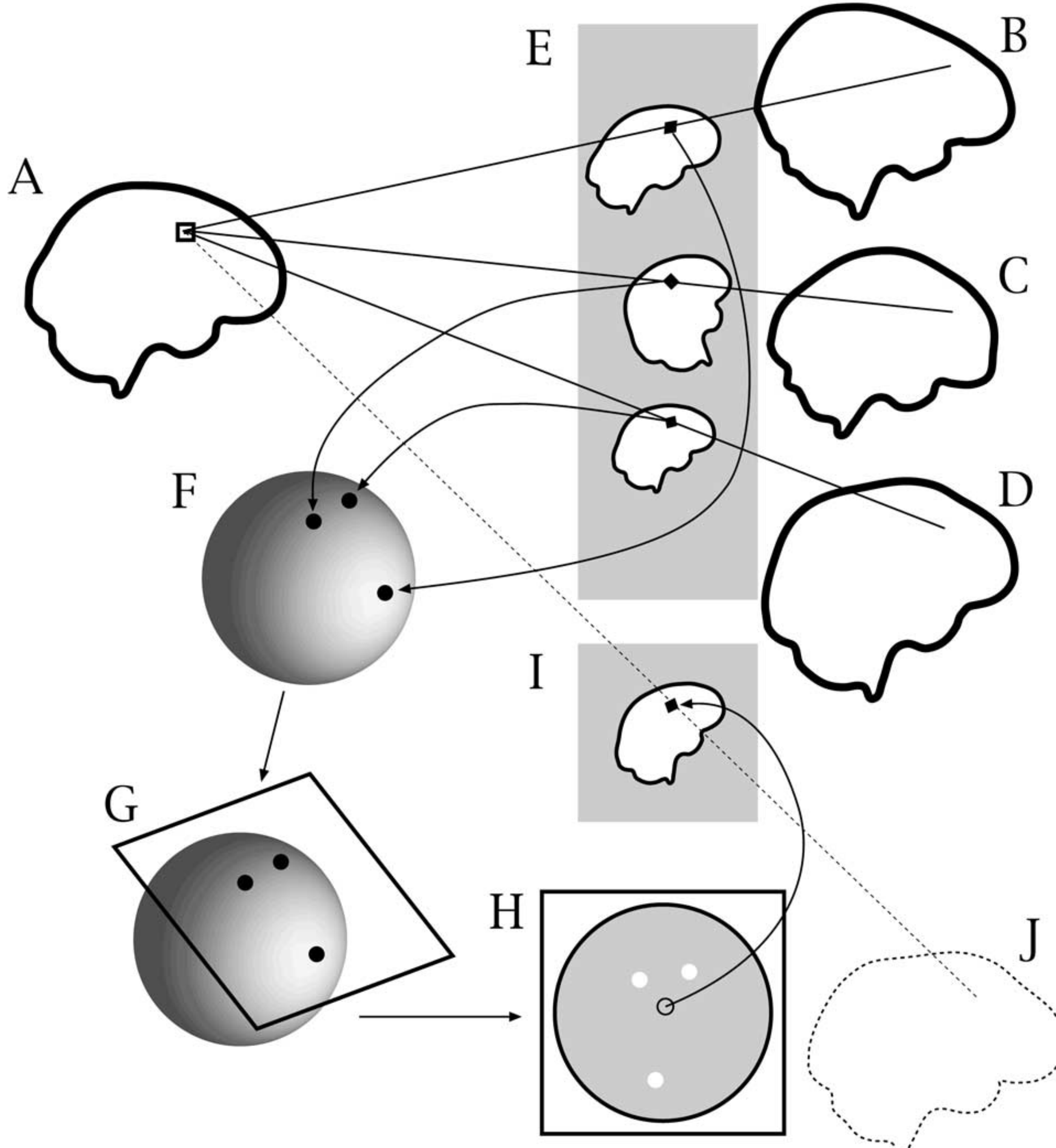


White circles show the locations of black ones after lifting into the tangent plane (a 2D Euclid space)

The white cross can be dropped back onto the sphere (Exp)

The new estimated mean has been rotated to the point of the sphere closest to you

The white X is located at the tangent point, indicating convergence


A
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Distance metrics

- Different measurements can require alternative concepts of distance. E.g.
  - Distances between angles
  - Distances within manifolds
  - Distances between matrices
  - Distances between special types of matrices...
- A metric satisfies

$$d(x,y) = d(y,x)$$

$$d(x,y) \geq 0, \quad d(x,y) = 0 \Rightarrow x = y$$

$$d(x,y) + d(y,z) \geq d(x,z)$$

Distance metrics

- Further (optional) properties may be desirable
  - Consider angles again, the distance should not be affected by rotating a pair of points by an equal amount
  - For rotation matrices we might want invariance to pre- and post-rotation
  - d(PAQ, PBQ) = d(A, B) for rotations A, B, P and Q
- For Jacobians, arbitrary choice of initial atlas and invariance to change of coordinates implies same

Matrix Lie Groups

- Rotation matrices are a compact group – they have a bi-invariant metric (Moakher)
- Jacobians are not, but are a semi-simple Lie group, with a bi-invariant pseudo-metric

Table 1
Some three dimensional geometric constraints, their associated Lie groups and tangent space parameterizations

Constraint	Lie group	Parameters	Tangent space parameterization
3D linear	$GL(3)^+$	9	$\begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{bmatrix}$
3D volume preserving	SL(3)	6	$\begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & -p_1-p_5 \end{bmatrix}$
3D conformal	$O(3)^+$	4	$\begin{bmatrix} p_1 & p_2 & -p_3 \\ -p_2 & p_1 & p_4 \\ p_3 & -p_4 & p_1 \end{bmatrix}$
3D rotations	SO(3)	3	$\begin{bmatrix} 0 & p_1 & -p_2 \\ -p_1 & 0 & p_3 \\ p_2 & -p_3 & 0 \end{bmatrix}$
2D linear	$GL(2)^+$	4	$\begin{bmatrix} p_1 & p_2 & 0 \\ p_3 & p_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
2D area preserving	SL(2)	3	$\begin{bmatrix} p_1 & p_2 & 0 \\ p_3 & -p_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
2D conformal	$O(2)^+$	2	$\begin{bmatrix} p_1 & p_2 & 0 \\ -p_2 & p_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
2D rotations	SO(2)	1	$\begin{bmatrix} 0 & p_1 & 0 \\ -p_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Semi-Riemannian manifolds

- A pseudo-metric can be negative
  - d(x,y) can be zero for x not equal to y
  - The Frechet mean is not well defined
- The Karcher mean (zero net velocity in tangent plane) is still well defined, and be found with an iterative proc

$$M_k = \exp\left(\frac{1}{N}\sum_{i=1}^{N} \exp^{-1}(A_i * M_{k-1}^{-1})\right) M_{k-1},$$

$$o = -\frac{1}{N}\sum_{i=1}^{N} \exp^{-1}(A_i * \mathrm{Mean}^{-1})$$

Analysis of deviations from mean

- $X_i = \mathrm{logm}(J_i M^{-1})$
  - Analogous to vector deviation from mean: j-m
  - Analogous to logarithmic deviation of positive scalars from geometric mean: log(j/m)
  - Deviation in the tangent plane at the mean
    - But note $||X_i||$ is not a distance
    - Recall only a pseudo-metric is available
- Multivariate statistics on Xi
  - Hotelling T-square test for comparing two groups
  - Wilks Lambda for more general regression models

Further issues

- Existence (Cartan decomposition)
- Analysis of “deviations” from mean
  – Not distances: how much of a problem is this?
- Removal of global pose
- Dealing with translations or perspective terms
- Surface deformations

Related work

- Diffusion tensors also on Riemannian manifold
  - An affine-invariant metric (and mean) can be found
  - Also, computationally trivial log-Euclidean metric
- Diffeomorphisms
  - Exp from velocity field to displacement field
  - Allows more sensible interpolation or extrapolation
  - Applications to e.g. motion models

References

- Further references, not included in Woods (2003)
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