Matrix Inversion Identities

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Two simple matrix identities are derived, these are then used to get expressions for the inverse of \((A + BCD)\). The expressions are variously known as the ‘Matrix Inversion Lemma’ or ‘Sherman-Morrison-Woodbury Identity’.

The derivation in these slides is taken from Henderson and Searle [1]. An alternative derivation, leading to a generalised expression, can be found in Tylavsky and Sohie [2].

Two special case results are mentioned, as they are useful in relating the Kalman-gain form and Information form of the Kalman Filter.
Identity 1

\[(I + P)^{-1} = (I + P)^{-1}(I + P - P)\]
\[= I - (I + P)^{-1}P\] (1)
Identity 2

\[ P + PQP = P(I + QP) = (I + PQ)P \]
\[ (I + PQ)^{-1}P = P(I + QP)^{-1} \] (2)
Matrix Inversion Lemma - step 1

For invertible \( A \), but general (possibly rectangular) \( B, C, \) and \( D \):

\[
(A + BCD)^{-1} = \left( A \left[ I + A^{-1}BCD \right] \right)^{-1}
\]

\[
= \left[ I + A^{-1}BCD \right]^{-1} A^{-1}
\]

\[
= \left[ I - \left( I + A^{-1}BCD \right)^{-1} A^{-1}BCD \right] A^{-1} \quad \text{Using (1)}
\]

\[
= A^{-1} - (I + A^{-1}BCD)^{-1} A^{-1}BCDA^{-1}
\]
Matrix Inversion Lemma - step 2

Repeatedly using (2) in sequence now produces:

\[(A + BCD)^{-1} = A^{-1} - (I + A^{-1}BCD)^{-1}A^{-1}BCDA^{-1}\] (3)
\[= A^{-1} - A^{-1}(I + BCDA^{-1})^{-1}BCDA^{-1}\] (4)
\[= A^{-1} - A^{-1}B(I + CDA^{-1}B)^{-1}CDA^{-1}\] (5)
\[= A^{-1} - A^{-1}BC(I + DA^{-1}BC)^{-1}DA^{-1}\] (6)
\[= A^{-1} - A^{-1}BCD(I + A^{-1}BCD)^{-1}A^{-1}\] (7)
\[= A^{-1} - A^{-1}BCDA^{-1}(I + BCDA^{-1})^{-1}\] (8)

(note that the order ABCD is maintained, ignoring the other parts of the expressions)
Matrix Inversion Lemma - special case

If C is also invertible, from (5):

\[(A + BCD)^{-1} = A^{-1} - A^{-1}B(I + CDA^{-1}B)^{-1}CDA^{-1} \]
\[= A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \quad (9)\]

which is a commonly used variant (for example applicable to the Kalman Filter covariance, in the ‘correction’ step of the filter).
Another related special case

A very similar use of (2) gives:

\[(A + BCD)^{-1} BC = A^{-1}(I + BCDA^{-1})^{-1} BC\]
\[= A^{-1}B(I + CDA^{-1}B)^{-1}C\]

and for invertible C:

\[(10)\]
\[= A^{-1}B(C^{-1} + DA^{-1}B)^{-1}\]
\[(11)\]

which is useful in converting between Kalman-gain and Information forms of the Kalman Filter state-estimate ‘correction’ step.
Bibliography
