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Itô and Stratonovich integrals on compound renewal processes: the normal/Poisson case

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ABSTRACT

Continuous-time random walks, or compound renewal processes, are pure-jump stochastic processes with several applications in insurance, finance, economics and physics. Based on heuristic considerations, a definition is given for stochastic integrals driven by continuous-time random walks, which includes the Itô and Stratonovich cases. It is then shown how the definition can be used to compute these two stochastic integrals by means of Monte Carlo simulations. Our example is based on the normal compound Poisson process, which in the diffusive limit converges to the Wiener process.

1. Introduction

1.1. The continuous-time random walk

The continuous-time random walk (CTRW) is a pure-jump stochastic process. It has been introduced by Montroll and Weiss in physics as a model for standard and anomalous diffusion when the residence time in a site is much greater than the jump time [1]. Shlesinger wrote a review paper that greatly contributed to popularize CTRWs [2]. More recently,

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theoretical and empirical studies on CTRWs have been discussed by Metzler and Klafter [3,4] and by a co-author of the present paper [5]. In a CTRW, if $x(t)$ denotes the position of a diffusing particle at time t, ξ_i denotes a random jump occurring at a random time t_i and $\tau_i = t_i - t_{i-1}$ is the inter-jump waiting time (also known as duration, interarrival interval or sojourn time), one has

$$x(t) = \sum_{i=1}^{n(t)} \xi_i, \tag{1}$$

where $t_0 = 0$, $x(0) = 0$ and $n(t)$ is a counting random process giving the number of jumps occurred up to time t. Throughout this paper, jumps, ξ_i, and waiting times, τ_i, are independent and identically distributed (i.i.d.) random variables; moreover, they are mutually independent random variables, so that the joint probability density $\varphi(\xi,\tau)$ can be factorized in terms of the marginal probability densities for jumps $w(\xi)$ and waiting times $\psi(\tau)$: $\varphi(\xi,\tau) = w(\xi)\psi(\tau)$; this is called the uncoupled case. Depending on their interpretation, jumps may be positive random variables or may assume any real value. They can also be vectors. On the other side, waiting times are positive random variables. Eq. (1) means that a CTRW is a random sum of independent random variables. The time process

$$t_n = \sum_{i=1}^{n} \tau_i, \quad t_0 = 0, \tag{2}$$

is a renewal point process. Therefore, CTRWs can be seen as compound renewal processes [6–8]. The existence of uncoupled CTRWs can be proved, based on the corresponding theorems of existence for renewal processes and discrete-time random walks. Càdlàg (right-continuous with left limit) realizations of CTRWs can be easily and exactly generated by Monte Carlo simulations and drawn. This is illustrated in Fig. 1. Uncoupled CTRWs are Markovian if and only if the waiting time distribution is exponential, meaning that $\psi(\tau) = \lambda \exp(-\lambda\tau)$ [9,10]. General uncoupled CTRWs belong to the class of semi-Markov processes [10,11]. For semi-Markov processes, it is possible to write an integral equation for the probability density $p(x,t)$ of finding the diffusing particle in position x at time t – the so-called Montroll–Weiss equation; this is done in terms of the marginal probability densities of waiting times $\psi(\tau)$ and of jumps $w(\xi)$:

$$p(x,t) = \Psi(t)\delta(x) + \int_{-\infty}^{+\infty} w(x-x') \int_0^t \psi(t-t') p(x',t')\,dt'\,dx', \tag{3}$$

where $\Psi(t) = 1 - \int_0^t \psi(u)\,du$ is known as survival function or complementary cumulative distribution function for waiting times. The solution of Eq. (3) can be written in terms of $P(n,t)$, the probability distribution function of the counting process $n(t)$, and $w^{*n}(x)$, the n-fold convolution of $w(\xi)$, as

$$p(x,t) = \sum_{n=0}^{\infty} P(n,t) w^{*n}(x). \tag{4}$$

This result can be derived from Eq. (3) using the Fourier and Laplace transforms, a method described in several papers, including the original one by Montroll and Weiss [1]. However, Eq. (4) can also be derived by direct probabilistic considerations. Indeed, Eq. (1) is a random sum of random i.i.d. variables. Position x can be reached at time t with either 0 or 1 or more jumps. The probability of reaching position x at time t in exactly n jumps is $P(n,t)w^{*n}(x)$. Eq. (4) follows given that these events are mutually exclusive. Note that $P(0,t)w^{*0}(x)$ coincides with the singular term $\Psi(t)\delta(x)$, meaning that the distribution function for x has a jump at position $x = 0$ of width $\Psi(t)$.

CTRWs with exponential waiting times (also called compound Poisson processes, as in this case $P(n,t) = \exp(-\lambda t)(\lambda t)^n/n!$) are not only Markovian, but they are also Lévy processes. This means that they have independent and time-homogeneous

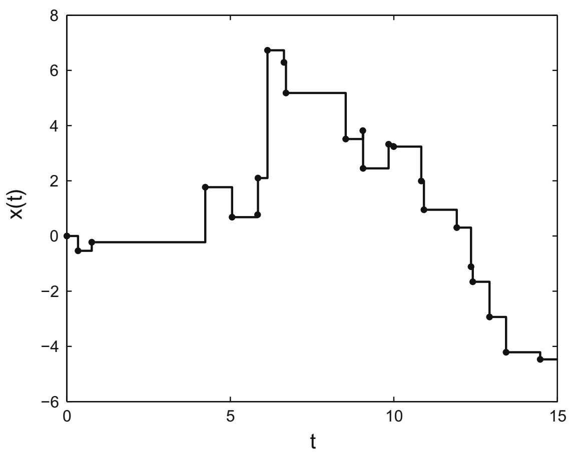

Fig. 1. Realization of a CTRW with exponentially distributed waiting times ($\lambda = 1$) and standard normally distributed jumps ($\mu = 0$ and $\sigma = 1$).

(stationary) increments. In this case, as a consequence of infinite divisibility and Kolmogorov's representation theorem, $p(x,t)$ (actually, $p(x,1)$) fully characterizes the stochastic process defined by Eq. (1) [12–14]. In general, without further dynamic specifications, the solution of Eq. (3) is not enough to fully characterize a stochastic process.

1.2. CTRWs in insurance, finance and economics

CTRWs have natural interpretations in insurance and finance theory. They can also be used in the theory of economic growth.

In ruin theory for insurance companies, the jumps ξ_i are interpreted as claims and they are positive random variables; t_i is the instant at which the ith claim is paid [15].

In finance theory, if $S(t)$ is the price of an asset at time t and S_0 is the price of the same asset at a previous reference time $t_0 = 0$, then $x(t) = \log(S(t)/S_0)$ represents the log-return (or log-price) at time t. In regulated markets using a continuous double-auction trading mechanism, such as stock markets, prices vary at random times t_i, when a trade takes place, and $\xi_i = x(t_i) - x(t_{i-1}) = \log(S(t_i)/S(t_{i-1}))$ is the tick-by-tick log-return, whereas $\tau_i = t_i - t_{i-1}$ is the intertrade duration; for more details, see Ref. [5, and references therein].

In the theory of economic growth, ξ_i represents a growth shock, $x(t)$ is the logarithm of the size for a firm or of the wealth for an individual and τ_i is the time interval between two consecutive growth shocks; again, see [5, and references therein].

1.3. Motivation for the study of stochastic integrals driven by CTRWs

Given the wide range of applications of CTRWs [3–5,16–18], it becomes important to study diffusive stochastic differential equations where noise is defined in terms of CTRWs:

$$dz = a(z,t)dt + b(z,t)dx, \tag{5}$$

where $z(x,t)$ is the unknown random function, $a(z,t)$ and $b(z,t)$ are known functions of z and time t, and $x(t)$ represents the CTRW with respect to which stochastic integrals are defined. In order to give a rigorous meaning to such an expression, some constraints on the properties of CTRWs are necessary. In a recent paper, the theory has been discussed for stochastic integration on time-homogeneous (stationary) CTRWs: the so-called compound Poisson processes (CPPs) [19]. Although the theory reported in Ref. [19] was already well known by mathematicians and has been used in finance for option pricing [20] since 1976, that paper contains useful material and is written in a way clear and appealing for physicists. The theory has been further discussed in Ref. [21]. Here we present a summary of that theory.

2. Stochastic integrals

In Ref. [19], the stochastic integral is not defined explicitly for a CTRW. However, starting from the fact that sample paths of a CTRW can be represented by step functions, it is possible to give an explicit formula.

2.1. Definition

For the definition of the stochastic integral

$$J(t) = \int_0^t G(x(s))dx(s), \tag{6}$$

where $x(t)$ is defined by Eq. (1) and $G(x)$ is a function defined in a suitable space, some heuristic manipulations are useful. Eq. (1) can be written in terms of Heaviside's function $\theta(t)$, which is 0 for $t < 0$ and 1 for $t \geqslant 0$:

$$x(t) = \sum_{i=1}^{n(t)} \xi_i \theta(t - t_i). \tag{7}$$

Using the fact that the "derivative" of Heaviside's θ function $\theta(t - t_i)$ is Dirac's δ function $\delta(t - t_i)$, one can write

$$dx(t) = \sum_{i=1}^{n(t)} \xi_i \delta(t - t_i)\, dt. \tag{8}$$

Note that $\delta(t)$ is not a function, but rather a distribution in the sense of Sobolev and Schwartz [22]. Replacing Eq. (8) in Eq. (6) and using the properties of Dirac's δ function, one gets

$$J(t) := \sum_{i=1}^{n(t)} G(x(t_i))\xi_i. \tag{9}$$

However, in order to define an integral à la Itô, it is necessary to make the integrand $G(x)$ statistically independent of the increment ξ_i and replace $G(x(t_i))$ in Eq. (9) with $G(x(t_i^-)) = G(x(t_{i-1}))$. This leads to the definition

$$I(t) := \int_0^t G(x(s^-))dx(s) = \sum_{i=1}^{n(t)} G(x(t_i^-))\xi_i = \sum_{i=1}^{n(t)} G(x(t_{i-1}))\xi_i; \tag{10}$$

with such a choice, the integrand becomes non-anticipating. An elementary introduction to the concept of a non-anticipating function can be found in Ref. [23]. A great advantage of Eq. (10) is that it can be easily implemented by means of Monte Carlo simulations, as will be shown in the next section. However, before that, it is useful to remark that one can similarly define the so-called Stratonovich integral

$$S(t) := \int_0^t G(x(s_{1/2}))dx(s) = \sum_{i=1}^{n(t)} G\left(\frac{x(t_i^-) + x(t_i)}{2}\right)\xi_i = \sum_{i=1}^{n(t)} G\left(\frac{x(t_{i-1}) + x(t_i)}{2}\right)\xi_i, \tag{11}$$

and, indeed, a full class of stochastic integrals,

$$J_\alpha(t) := \int_0^t G(x(s_\alpha))dx(s) = \sum_{i=1}^{n(t)} G((1-\alpha)x(t_i^-) + \alpha x(t_i))\xi_i = \sum_{i=1}^{n(t)} G((1-\alpha)x(t_{i-1}) + \alpha x(t_i))\xi_i, \tag{12}$$

where $\alpha \in [0,1]$, so that $I(t) = J_0(t),\ S(t) = J_{1/2}(t)$, and $J(t) = J_1(t)$.

3. Simulations

Suppose one desires to compute the value $x(t)$; it is then sufficient to generate a sequence of $n(t)+1$ i.i.d. waiting times τ_i until their sum is greater than t. Then the last waiting time can be discarded and $n(t)$ i.i.d. jumps ξ_i can be generated. Their sum is the desired value of $x(t)$. Based on Eqs. (1) and (2), this algorithm was used to generate Fig. 1.

Similarly, an algorithm based on Eqs. (10), (11) or (12) can be implemented by generating a sequence of $n(t)+1$ i.i.d. waiting times τ_i until their sum is greater than t. Then after generating $n(t)$ i.i.d. distributed jumps ξ_i their values can be multiplied by $G(x(t_{i-1}))$, $G((x(t_{i-1}) + x(t_i))/2)$, or $G((1-\alpha)x(t_{i-1}) + \alpha x(t_i))$, respectively, and the results of these multiplications can be summed to obtain $I(t)$, $S(t)$, or $J_\alpha(t)$. In Fig. 2, a Monte Carlo generated histogram for $I(t) = \int_0^t x(s^-)\,dx(s)$ (with $G(x) = x$) is given, where $t = 100$ and $x(t)$ is a normal compound Poisson process (NCPP). The simulated NCPP has exponentially distributed waiting times with $\lambda = 1$ and normally distributed jumps $w(\xi) = \exp[-(\xi-\mu)^2/(2\sigma)]/\sqrt{2\pi\sigma}$ with $\mu = 0$ and $\sigma = 1$. For a general NCPP, the probability density of finding the value x at time t is given by

$$p(x,t) = \exp(-\lambda t)\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \frac{1}{\sqrt{2\pi n\sigma}} \exp\left[-\frac{(x-n\mu)^2}{2n\sigma}\right]. \tag{13}$$

The NCPP approximates the Bachelier–Wiener process $W(t)$ for $\lambda \to \infty$ and $\sigma \to 0$ with $\lambda\sigma^2 = \sigma_W^2$ [19]; therefore, if $x(t)$ is an NCPP, the integral in Eq. (10) is an approximation of the usual Itô integral driven by a Wiener process, $I_W(t)$. This point is illustrated in Fig. 3, where the histogram of 50,000 values of $I(1) = \int_0^1 x(s^-)\,dx(s)$ when $\lambda = 10{,}000$ and $\sigma = 1/100$ is compared to the analytic expression of the probability density for $\sigma_W = 1$ when $I_W(t) = \int_0^t W(s^-)\,dW(s) = (W^2(t) - t)/2$ and $t = 1$. Under the same hypotheses, the integral in Eq. (11) converges to the usual Stratonovich integral. This is shown in Fig. 4, where the histogram of 50,000 values of $S(1) = \int_0^1 x(s_{1/2})\,dx(s)$ when $\lambda = 10{,}000$ and $\sigma = 1/100$ is compared to the analytic expression of the probability density for $\sigma_W = 1$ when $S_W(t) = \int_0^t W(s_{1/2})\,dW(s) = W^2(t)/2$ and for $t = 1$.

The agreement between the Monte Carlo histogram and the analytic formula is excellent. Even if a detailed study of convergence properties and bounds is beyond the scope of the present paper, the results of these Monte Carlo simulations lead to conjecture that the integrals $J_\alpha(t)$ weakly converge to the corresponding integrals driven by the Bachelier–Wiener process, $W(t)$, when the pure-jump process $x(t)$ converges to $W(t)$.

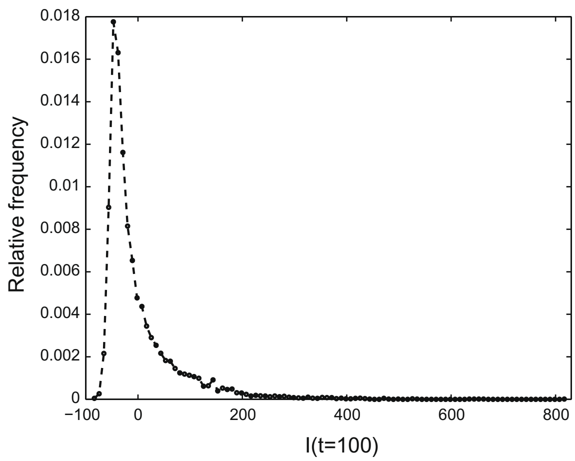

Fig. 2. Histogram of the integral $I(t) = \int_0^t x(s^-)\,dx(s)$ for exponentially distributed waiting times ($\lambda = 1$) and standard normally distributed jumps ($\mu = 0$ and $\sigma = 1$) and for $t = 100$. The circles represent the results of 10,000 independent realizations of the integral. The dashed line is plotted to guide the eye.

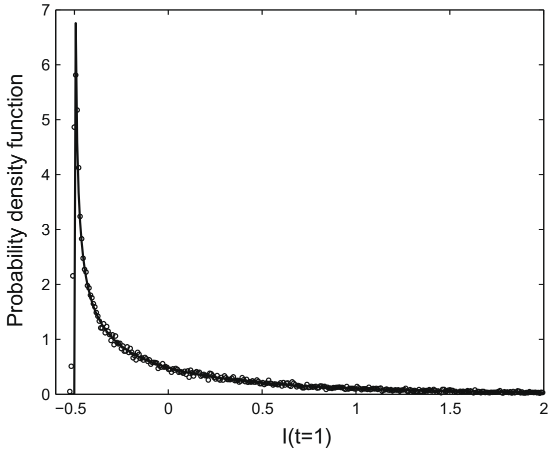

Fig. 3. Comparison between the empirical probability density from Monte Carlo calculation (circles) of $I(t) = \int_0^t x(s^-)\,dx(s)$ and the analytic probability density for the Itô integral (solid line) $I_W(t) = \int_0^t W(s^-)\,dW(s) = (W^2(t) - t)/2$, where $W(t)$ is the Bachelier–Wiener process and $x(t)$ is a NCPP with $\lambda = 10,000,\ \mu = 0,\ \sigma = 1/100$, yielding $\sigma_W = 1$ for the limiting Bachelier–Wiener process. In this plot $t = 1$ and I_W has the probability density $p(I_W) = 2\exp[-(2I_W + 1)/2]/\sqrt{2\pi(2I_W + 1)}$.

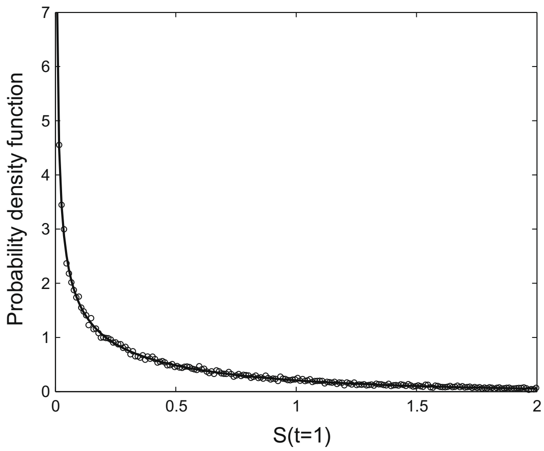

Fig. 4. Comparison between the empirical probability density from Monte Carlo calculation (circles) of $S(t) = \int_0^t x(s_{1/2})\,dx(s)$ and the analytic probability density for the Stratonovich integral (solid line) $S_W(t) = \int_0^t W(s_{1/2})\,dW(s) = W^2(t)/2$, where $W(t)$ is the Bachelier–Wiener process and $x(t)$ is a NCPP with $\lambda = 10,000,\ \mu = 0,\ \sigma = 1/100$, yielding $\sigma_W = 1$ for the limiting Bachelier–Wiener process. In this plot $t = 1$ and S_W has the probability density $p(S_W) = 2\exp(-S_W)/\sqrt{4\pi S_W}$.

4. Conclusions

This paper gives a mathematically rigorous definition of stochastic integrals driven by CTRWs, see Eqs. (10)–(12). These relations are easily used in Monte Carlo calculations of stochastic integrals. In Section 3 it is shown how to use Monte Carlo simulations of the NCPP to effectively approximate the usual Itô integral based on the Bachelier–Wiener process.

While the NCPP is both a Markov and a Lévy process, general uncoupled CTRWs do not share these two properties, but they belong to the class of semi-Markov processes. It is known that by adding a new random operational time it is often possible to generate a bivariate Markov process $(s(.), x(.))$, but this line of research has not yet been followed-up in the literature in the context of semi-Markov stochastic integrals. Typically, $s(.)$ is the random time span between the present time t and the time of the next jump; in this case semi-martingale methods should be applicable. Another possibility to justify semi-martingale techniques is to consider the case where the jumps of a CTRW have zero mean, since in this particular case a CTRW is a martingale; as a consequence, the Itô integral is also a martingale [21].

Future work will focus on more general uncoupled and coupled CTRWs where jumps and waiting times follow fat-tailed distributions [5,21,24–27]. If $x(t)$ is a suitable uncoupled CTRW, in the diffusive limit its probability density function $p(x,t)$ converges to the solution of the space-time fractional diffusion equation; the corresponding stochastic differential equation is given by Eq. (5) with $a = 0$ and $b = D$. Methods from the theory of semi-martingales might be used in order to prove the convergence of the stochastic integral in the presence of infinitely many jumps in compact time intervals [28].

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