Semantics ex Proof and Refutation
(Extended Abstract)

A logic may be regarded as a consequence relation between collections of formulas; typically characterized either through provability in a reasoning system (e.g., natural deduction) or through entailment in a semantics. A standard task of the logician to show that two such characterizations are equivalent: a system is *sound* with respect to a semantics if provability implies entailment; it is *complete* if entailment implies provability. Taking provability and entailment as predicates, soundness claims that provability is a subset of entailment and completeness claims that entailment is a subset of provability. Hence, soundness and completeness concerns the *extensional* equivalence of two characterizations of a logic.

Since rules of provability and clauses of the semantics *define* the connectives of the logic, one expects the proof-theoretic and semantic views of logic to be reasonably close. Indeed, one may expect not only extensional equivalence to hold but *intensional* equivalence (i.e., that the two relations *behave* the same); for example, the semantic clause for conjunction precisely captures the introduction rule for conjunction *and* its invertibility. Hence, to prove soundness and completeness one may show that entailment and provability *simulate* each other.

Nonetheless, there is a distinction between entailment and provability that renders the problem of behavioural equivalence complex. A clause for a connective is *definitional* in that it has an *iff* structure; in contrast, a rule is *implicational* in that the conclusion holds when the premisses do, but the converse is not necessarily true. That provability is implicational is to say that it concerns *deduction*:

$$\frac{\text{Established Premiss}_1 \quad \dots \quad \text{Established Premiss}_n}{\text{Established Conclusion}} \Downarrow$$

The systematic use of mathematical techniques to determine the forms of valid deductive argument defines *deductive logic*. Taking validity to mean entailment, soundness concerns the correctness of the rules in that they preserve validity. To understand completeness analogously, one considers the contrapositive: a logic is complete if non-provability implies non-entailment. The significance of this view is that proofs and refutations become equal citizens when considering the relationship between provability and entailment for a logic.

To witness non-provability one must show that any attempt at constructing a proof fails, which may be done by applying proof rules as reduction operators, taking a putative conclusion to some set of sufficient premisses. In this case rules do not determine deductions but rather *reductions*,

$$\frac{\text{Sufficient Premiss}_1 \quad \ldots \quad \text{Sufficient Premiss}_n}{\text{Putative Conclusion}} \Uparrow$$

The study of reduction operators, the dual of deductive logic, is *reductive logic.* The space of reductions of a putative conclusion is typically larger than its space of proofs, including also failed searches.

In contrast to deductive logic, reductive logic currently has no uniform mathematical foundation; though there has been substantial research done on the reductive logic of classical and intuitionistic calculi (e.g., Pym and Ritter [5]). Yet, reductive logic captures the question of when there is a proof in the logic (as opposed to what proofs are), which, by the above argument, renders it of central importance for understanding the relationship between proof and semantics. More practically, reductive logic offers a principled way of *deriving* a semantics from a reasoning system.

Given a logic with consequence defined by provability (e.g., as is often the case for substructural logics — see, for example, Read [6]), one may generate a semantics by designing an entailment to mimic the behaviour of the reduction operators. In model-theoretic semantics, entailment can be studied directly (i.e., avoiding truth-in-a-model) by working with *eigenworlds* w — generic representatives of worlds — in a (classical) *meta*-logic rather than actual worlds in an actual model; for example, enforcing the following simulation means that the connective $\land$ may be modelled by classical conjunction:

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \land - \text{rule} \quad \sim \quad \frac{\Sigma, w \Vdash \Gamma : w \Vdash \phi, \Pi \quad \Sigma, w \Vdash \Gamma : w \Vdash \psi, \Pi}{\Sigma, w \Vdash \Gamma : w \Vdash \phi \land \psi, \Pi} \land\text{-clause}$$

More generally, a semantics for a logic $\mathcal{L}$ can be determined from those of another logic $\mathcal{L}'$ for which one already has a sound and complete semantics by encapsulating how $\mathcal{L}$'s connectives diverge from those of $\mathcal{L}'$. A case-study is the relationship between intuitionistic logic (IL) and classical logic (CL) — typically, $\mathcal{L}'$ is CL.

Dummett's [1] multiple-conclusioned sequent calculus DJ shows that, except for implication, the connectives of IL behave as their corresponding connectives of CL. Consequently, each connective (including disjunction) may be modelled by the corresponding classical connective. In the remaining case, DJ does require sequents to be single-conclusioned,

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}$$

This condition is captured in the clause for implication in model-theoretic semantics by a guard that enforces a change world:

$w \Vdash \phi \to \psi$ if and only if, for any u, *if* $w \preceq u$, then $u \Vdash \phi$ implies $u \Vdash \psi$

Essentially, the $w \preceq u$ condition preserves the context Γ through persistence — if $w \preceq u$, then $w \Vdash \Gamma$ implies $u \Vdash \Gamma$ — while ensuring not other assertion can interact with the new assumption ϕ. For example, the following *invertible* computation demonstrates why the law of the excluded middle fails for IL-entailment:

$$
\left.\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w \Vdash \Gamma, u \Vdash \Gamma, \phi : w \Vdash \phi, u \Vdash \bot}{w \Vdash \Gamma, u \Vdash \Gamma, u \Vdash \phi : w \Vdash \phi, u \Vdash \bot}\ \wedge\text{-clause}}{w \Vdash \Gamma, w \prec u, u \Vdash \phi : w \Vdash \phi, u \Vdash \bot}\ \text{persistance}}{w \Vdash \Gamma, w \prec u : u \Vdash \phi, (u \Vdash \phi \implies u \Vdash \bot)}}{w \Vdash \Gamma : w \Vdash \phi, (w \prec u \implies (u \Vdash \phi \implies u \Vdash \bot))}}{\dfrac{w \Vdash \Gamma : w \Vdash \phi, w \Vdash \phi \to \bot}{w \Vdash \Gamma : w \Vdash \phi \vee \phi \to \bot}\ \vee\text{-clause}}
\end{array}\right\}
\begin{array}{c}
\dfrac{\Gamma \models \phi}{\Gamma \models \phi \vee \phi \to \bot}\ \Uparrow \\[2ex]
\text{or} \\[2ex]
\dfrac{\Gamma, \phi \models \bot}{\Gamma \models \phi \vee \phi \to \bot}\ \Uparrow
\end{array}
$$

Of course, this analysis relies on a number of implicit design choices; for example, the idea of using an algebra to relativize truth is one such choice, which distinguishes model-theoretic and proof-theoretic semantics.

The connexion between Dummett's system and CL (as characterized by Gentzen's LK [7]) is the technology that delivers the above analysis; and, generally, the relationship between $\mathcal{L}$ and $\mathcal{L}'$. It can be made precise by means of *algebraic constraints* in the style of Harland's and Pym's *resource-distribution via boolean constraints* mechanism [3, 4]. The rule for IL's implication may be expressed as the LK rule for implication enriched with boolean variables x,

$$
\frac{\Gamma, \phi \cdot x \vdash \psi \cdot x, \Delta \cdot \bar{x}}{\Gamma \vdash \phi \to \psi, \Delta}
$$

An instance of the rule is given by an assignment on x according to which formulas assigned 1 are kept and formulas assigned 0 are removed. It is in capturing the effect of these constraints to the world algebra that one generates the guarding in the clause for implication in the model-theoretic semantics of IL. A general perspective of such algebraic constrains system in reductive logic have been discussed by Gheorghiu et al. [2].

To conclude, reductive logic is both a natural and useful paradigm for studying logic that can help understand the relationship between the two major characterization of consequence, provability and entailment. Furthermore, it can help derive the latter from the former by giving an understanding of precisely when provability holds, using algebraic constraints to relate the logic being studied to one that is well-understood: eigenworlds allow one to study entailment directly, and using the constraints to control their behaviour delivers the desired semantics. This may enable general and algorithmic techniques for studying the relationship between proof to be developed.

References

[1] Michael Dummett. *Elements of Intuitionism*, volume 39 of *Oxford Logic Guides*. Oxford University Press, 2000.

[2] Alexander V. Gheorghiu, Simon Docherty, and David J. Pym. Reductive Logic, Coalgebra, and Proof-search: A Perspective from Resource Semantics. In A. Palmigiano and M. Sadrzadeh, editors, *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, Springer Outstanding Contributions to Logic Series. Springer, 2021. to appear.

[3] James Harland and David J. Pym. Resource-distribution via Boolean Constraints. In *Proceedings of the International Conference on Automated Deduction — CADE-14*, pages 222–236. Springer, 1997.

[4] James Harland and David J. Pym. Resource-distribution via Boolean Constraints. *ACM Transactions on Computational Logic*, 4(1):56–90, 2003.

[5] David J. Pym and Eike Ritter. *Reductive Logic and Proof-search: Proof Theory, Semantics, and Control*. Oxford Logic Guides. Clarendon Press, 2004.

[6] Stephen Read. *Relevant Logic*. Basil Blackwell, 1988.

[7] M. E. Szabo, editor. *The Collected Papers of Gerhard Gentzen*. North-Holland Publishing Company, 1969.