


Pomsets with Boxes: Protection, Separation, and Locality in Concurrent Kleene Algebra

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Abstract

Concurrent Kleene Algebra is an elegant tool for equational reasoning about concurrent programs. An important feature of concurrent programs that is missing from CKA is the ability to restrict legal interleavings. To remedy this we extend the standard model of CKA, namely pomsets, with a new feature, called boxes, which can specify that part of the system is protected from outside interference. We study the algebraic properties of this new model. Another drawback of CKA is that the language used for expressing properties of programs is the same as that which is used to express programs themselves. This is often too restrictive for practical purposes. We provide a logic, ‘pomset logic’, that is an assertion language for specifying such properties, and which is interpreted on pomsets with boxes. In contrast with other approaches, this logic is not state-based, but rather characterizes the runtime behaviour of a program. We develop the basic metatheory for the relationship between pomset logic and CKA, including frame rules to support local reasoning, and illustrate this relationship with simple examples.

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Supplement Material Since proofs are omitted from this paper, we refer the inquisitive reader to its extended version, available on arXiv: [1910.14384](https://arxiv.org/abs/1910.14384). A formalization of Section 2 in Coq is available on github: [AtomicCKA](https://github.com/AtomicCKA).

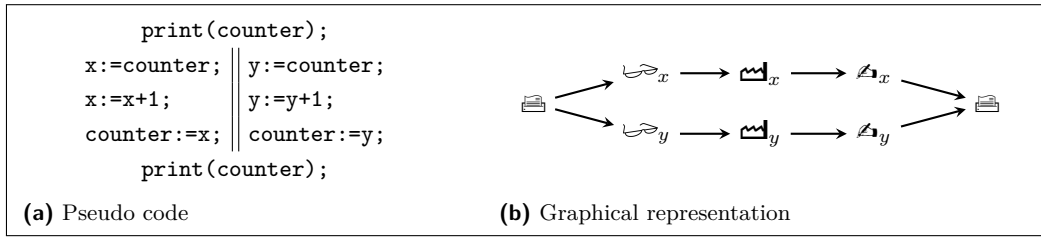
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1 Introduction

Concurrent Kleene Algebra (CKA) [11, 14, 15, 4] is an elegant tool for equational reasoning about concurrent programs. Its semantics is given in terms of pomsets languages; that is, sets of pomsets. Pomsets [8], also known as partial words [9], are a well-known model of concurrent behaviour, traditionally associated with runs in Petri nets [13, 4].

However, in CKA the language used for expressing properties of programs is the same as that which is used to express programs themselves. It is clear that this situation is not ideal for specifying and reasoning about properties of programs. Any language specifiable in CKA terms has bounded width (i.e., the number of processes in parallel; the size of a maximal independent set) and bounded depth (i.e., the number of alternations of parallel



■ **Figure 1** Distributed counter

and sequential compositions)[17]. However, many properties of interest — for example, safety properties — are satisfied by sets of pomsets with both unbounded width and unbounded depth.

In this paper, we provide a logic, ‘pomset logic’, that is an assertion language for specifying such properties. We develop the basic metatheory for the relationship between pomset logic and CKA and illustrate this relationship with simple examples. In addition, to the usual classical or intuitionistic connectives — both are possible — the logic includes connectives that characterize both sequential and parallel composition.

In addition, we note that CKA allows programs with every possible interleaving of parallel threads. However, to prove the correctness of such programs, some restrictions must be imposed on what are the legal interleavings. We provide a mechanism of ‘boxes’ for this purpose. Boxes identify protected parts of the system, so restricting the possible interleavings. From the outside, one may interact with the box as a whole, as if the program inside was atomic. On the other hand, it is not possible to interact with its individual components, as that would intuitively require opening the box. However, boxes can be nested, with this atomicity observation holding at each level. Pomset logic has context and box modalities that characterize this situation.

► **Note.** The term ‘Pomset logic’ has already been used in the literature by Retoré [25]. We feel that reusing that phrase does not introduce ambiguity, since the two frameworks arise in different contexts.

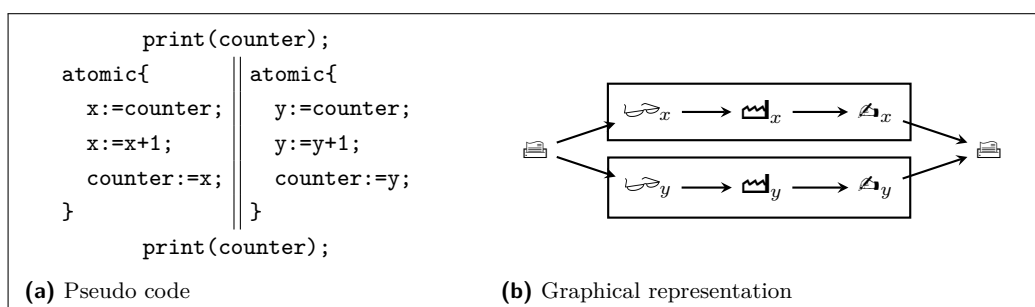
► **Example 1** (Running example: a distributed counter). We consider here a program where a counter is incremented in parallel by two processes. The intention is that the counter should be incremented twice, once by each process. However, to do so each process has to first load the contents of the counter, then compute the increment, and finally commit the result to memory. A naive implementation is presented in Figure 1a. Graphically, we represent the print instruction `print(counter)` by , the read instruction `x:=counter` by , the increment instruction `x:=x+1` by , and finally the write instruction `counter:=x` by . We thus represent the previous program as displayed in Figure 1b.

This program does not comply with our intended semantics, since the following run is possible:

$$\text{box} \longrightarrow \text{hook}_x \longrightarrow \text{hook}_y \longrightarrow \text{wavy}_x \longrightarrow \text{wavy}_y \longrightarrow \text{triangle}_x \longrightarrow \text{triangle}_y \longrightarrow \text{box}$$

The result is that the counter has been incremented by one. We can identify a subset of instructions that indicate there is a fault: the problem is that both read instructions happened before both write instructions; i.e.,

$$\begin{array}{ccc} \text{hook}_x & \longrightarrow & \text{triangle}_x \\ \text{hook}_y & \longrightarrow & \text{triangle}_y \end{array}$$



■ **Figure 2** Distributed counter with atomic increment

To preclude this problematic behaviour, a simple solution is to make the sequence ‘read;compute;write’ *atomic*. This yields the program in Figure 2a. Diagrammatically, this can be represented by drawing solid boxes around the `atomic{}` blocks, as shown in Figure 2b. This paper shows how to make these ideas formal.

In Section 2, we extend pomsets with a new construct for protection, namely boxes. We provide a syntax for specifying such pomsets and characterize precisely its expressivity. This enables us, for example, to correctly represent the program from Example 1. We present a sound and complete axiomatization of these terms, with operators for boxing, sequential and parallel composition, and non-deterministic choice, as well as the constants *abort* and *skip*.

In Section 3, we introduce pomset logic. This logic comes in both classical and intuitionistic variants. In addition to the usual classical or intuitionistic connectives, this logic includes connectives corresponding to each of sequential and parallel composition. These two classes of connectives are combined to give the overall logics in the same way as the additives and multiplicatives of BI (bunched implications logic) [21, 1, 23]. Just as in BI and its associated separation logics [21, 12, 26], pomset logic has both classical and intuitionistic variants. Pomset logics also include modalities that characterize, respectively, protection, and locality. These correspondences are made precise by van Benthem–Hennessy–Milner-type theorems asserting that two programs are (operationally) equivalent iff they satisfy the same formulae. We obtain a number of such correspondences, for several variants of our framework. In contrast to Hennessy–Milner logic, however, pomset logic is a logic of *behaviours* rather than of states and transitions.

In Section 4, we investigate local reasoning principles for our logic of program behaviours. We showcase the possibilities of our framework on an example. We conclude by briefly discussing future work in Section 5.

2 Algebra of Pomsets with Boxes

In this section, we define our semantic model, and the corresponding syntax. We characterize the expressivity of the syntax, and axiomatize its equational theory.

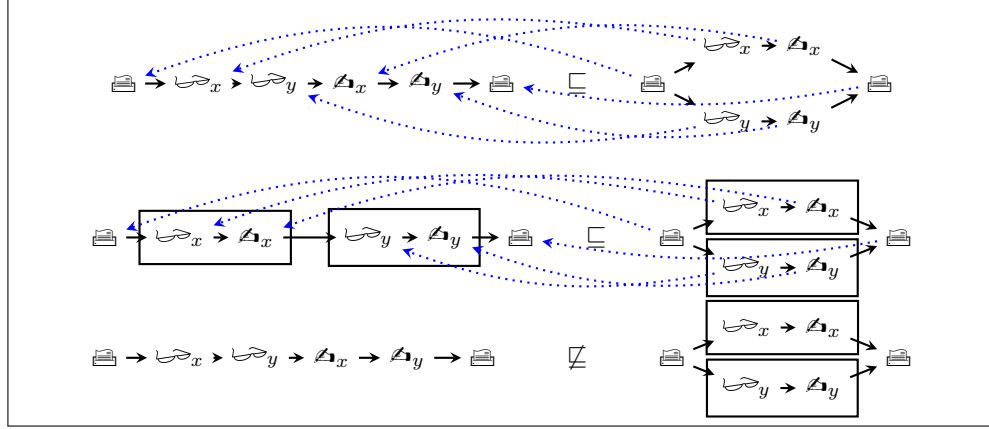
Throughout this paper, we will use Σ to denote a given set of atomic actions.

2.1 Pomsets with boxes

2.1.1 Definitions and elementary properties

► **Definition 2** (Poset with boxes). *A poset with boxes is a tuple $P := \langle \mathcal{E}_P, \leq_P, \lambda_P, \mathcal{B}_P \rangle$, where \mathcal{E}_P is a finite set of events; $\leq_P \subseteq \mathcal{E}_P \times \mathcal{E}_P$ is a partial order; $\lambda_P : \mathcal{E}_P \rightarrow \Sigma$ is a labelling*

■ **Figure 3** Poset subsumption



function; $\mathcal{B}_P \subseteq \mathcal{P}(\mathcal{E}_P)$ is a set of boxes, such that $\emptyset \notin \mathcal{B}_P$.

The partial order should be viewed as a set of necessary dependencies: in any legal scheduling of the pomset, these dependencies have to be satisfied. We therefore consider that a stronger ordering — that is, one containing more pairs — yields a smaller pomset. The intuition is that the set of legal schedulings of the smaller pomset is contained in that of the larger one. The boxes are meant to further restrict the legal schedulings: no event from outside a box may be interleaved between the events inside the box. Subsequently, a pomset with more boxes is smaller than one with less boxes. This ordering between pomsets with boxes is formalized by the notion of homomorphism:

► **Definition 3** (Poset morphisms). A (poset with boxes) homomorphism is a map between event-sets that is bijective, label respecting, order preserving, and box preserving. In other words, a map $\phi : \mathcal{E}_P \rightarrow \mathcal{E}_Q$ such that (i) ϕ is a bijection; (ii) $\lambda_Q \circ \phi = \lambda_P$; (iii) $\phi(\leq_P) \subseteq \leq_Q$; (iv) $\phi(\mathcal{B}_P) \subseteq \mathcal{B}_Q$. If in addition (iii) holds as an equality, ϕ is called order-reflecting. If on the other hand (iv) holds as an equality ϕ is box-reflecting. A homomorphism that is both order- and box-reflecting is a (poset with boxes) isomorphism.

In Figure 3 are some examples and a non-example of subsumption between posets. We introduce some notations. \mathbb{P}_Σ is the set of posets with boxes. If ϕ is a homomorphism from P to Q , we write $\phi : P \rightarrow Q$. If there exists such a homomorphism (respectively an isomorphism) from P to Q , we write $Q \sqsubseteq P$ (resp. $Q \cong P$).

► **Lemma 4.** \cong is an equivalence relation. \sqsubseteq is a partial order with respect to \cong .

► **Remark 5.** Note that the fact that \sqsubseteq is antisymmetric with respect to \cong relies on the finiteness of the posets considered here. Indeed, we can build infinite pomsets that are not isomorphic but have nevertheless homomorphism between them in both directions.

► **Definition 6** (Pomsets with boxes). Pomsets with boxes are equivalence classes of \cong . The set \mathbf{Pom}_Σ of pomsets with boxes is defined as $\mathbb{P}_\Sigma / \cong$.

We now define some elementary poset-building operations.

► **Definition 7** (Constants). Given a symbol $a \in \Sigma$, the atomic poset associated with a is defined as $\mathfrak{a} := \langle \{0\}, [0 \mapsto a], Id_{\{0\}}, \emptyset \rangle \in \mathbb{P}_\Sigma$. The empty poset is defined as $\mathfrak{e} := \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle \in \mathbb{P}_\Sigma$.

► **Remark 8.** For any poset $P \in \mathbb{P}_\Sigma$, $P \sqsubseteq \epsilon \Leftrightarrow P \sqsupseteq \epsilon \Leftrightarrow P \cong \epsilon$. This is because each of those relations imply there is a bijection between the events of P and $\mathcal{E}_\epsilon = \emptyset$. So we know that P has no events, and since boxes cannot be empty, P has no boxes either. Hence $P \cong \epsilon$.

► **Definition 9 (Compositions).** Let P, Q be two posets with boxes. The sequential composition $P \otimes Q$ and parallel composition $P \oplus Q$ are defined by:

$$\begin{aligned} P \otimes Q &:= \langle \mathcal{E}_P \uplus \mathcal{E}_Q, \leq_P \cup \leq_Q \cup (\mathcal{E}_P \times \mathcal{E}_Q), \lambda_P \sqcup \lambda_Q, \mathcal{B}_P \cup \mathcal{B}_Q \rangle \\ P \oplus Q &:= \langle \mathcal{E}_P \uplus \mathcal{E}_Q, \leq_P \cup \leq_Q, \lambda_P \sqcup \lambda_Q, \mathcal{B}_P \cup \mathcal{B}_Q \rangle, \end{aligned}$$

where the symbol \sqcup denotes the union of two functions; that is, given $f : A \rightarrow C$ and $g : B \rightarrow C$, the function $f \sqcup g : A \uplus B \rightarrow C$ associates $f(a)$ to $a \in A$ and $g(b)$ to $b \in B$.

Intuitively, $P \oplus Q$ consists of disjoint copies of P and Q side by side. $P \otimes Q$ also contains disjoint copies of P and Q , but also orders every event in P before any event in Q .

► **Definition 10 (Boxing).** Given a poset P its boxing is denoted by $[P]$ and is defined by: $[P] := \langle \mathcal{E}_P, \leq_P, \lambda_P, \mathcal{B}_P \cup \{\mathcal{E}_P\} \rangle$.

Boxing a pomset simply amounts to drawing a box around it.

In our running example, the pattern of interest is a subset of the events of the whole run. To capture this, we define the restriction of a poset to a subset of its events.

► **Definition 11 (Restriction, sub-poset).** For a given set of events $A \subseteq \mathcal{E}_P$, we define the restriction of P to A as $P \downarrow_A := \langle A, \leq_P \cap (A \times A), \lambda_P \downarrow_A, \mathcal{B}_P \cap \mathcal{P}(A) \rangle$. We say that P is a sub-poset of Q , and write $P \in Q$, if there is a set $A \subseteq \mathcal{E}_Q$ such that $P \cong Q \downarrow_A$.

Given a poset P , a set of events $A \subseteq \mathcal{E}_P$ is called:

- **nested** if for any box $\beta \in \mathcal{B}_P$ either $\beta \subseteq A$ or $A \cap \beta = \emptyset$;
- **prefix** if for any $e \in A$ and $f \notin A$ we have $e \leq_P f$; and
- **isolated** if for any $e \in A$ and $f \notin A$ we have $e \not\leq_P f$ and $f \not\leq_P e$.

These properties characterize sub-posets of particular interest to P . This is made explicit in the following observation:

► **Fact 12.** Given a poset P and a set of events $A \subseteq \mathcal{E}_P$:

- (i) A is prefix and nested iff $P \cong P \downarrow_A \otimes P \downarrow_{\bar{A}}$;
 - (ii) A is isolated and nested iff $P \cong P \downarrow_A \oplus P \downarrow_{\bar{A}}$.
- (Here \bar{A} denotes the complement of A relative to \mathcal{E}_P ; that is, $\bar{A} := \mathcal{E}_P \setminus A$.)

This fact is very useful as a way to ‘reverse-engineer’ how a poset was built.

2.1.2 Series–parallel pomsets

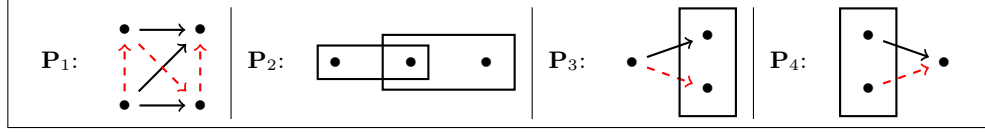
In the sequel, we will often restrict our attention to series–parallel pomsets. These are of particular interest since they are defined as those pomsets that can be generated from constants using the operators we have defined.

► **Definition 13 (Pomset terms, SP-Pomsets).** A (pomset) term is a syntactic expression generated from the following grammar: $s, t \in \text{SP}_\Sigma ::= 1 \mid a \mid s ; t \mid s \parallel t \mid [s]$. By convention ; binds tighter than \parallel . A term is interpreted as a poset as follows:

$$\begin{aligned} [a] &:= \mathfrak{a} & [1] &:= \epsilon & [[s]] &:= [[s]] \\ [s ; t] &:= [s] \otimes [t] & [s \parallel t] &:= [s] \oplus [t]. \end{aligned}$$

A pomset $[P]_{\cong}$ is called series–parallel (or SP for short) if it is the interpretation of some term; that is, $\exists s \in \text{SP}_\Sigma : [s] \cong P$.

■ **Figure 4** Forbidden patterns of SP-pomsets: dashed arrows (in red) are negated.



► **Example 14.** The program in Figure 1 of the running example corresponds to

$$\llbracket \boxed{\cdot}; (\hookrightarrow_x; \llbracket \cdot \rrbracket_x; \llbracket \cdot \rrbracket_x \parallel \hookrightarrow_y; \llbracket \cdot \rrbracket_y; \llbracket \cdot \rrbracket_y); \boxed{\cdot} \rrbracket.$$

The corrected program, from Figure 2, corresponds to

$$\llbracket \boxed{\cdot}; (\llbracket \hookrightarrow_x; \llbracket \cdot \rrbracket_x; \llbracket \cdot \rrbracket_x \rrbracket \parallel \llbracket \hookrightarrow_y; \llbracket \cdot \rrbracket_y; \llbracket \cdot \rrbracket_y \rrbracket); \boxed{\cdot} \rrbracket.$$

Finally, the problematic pattern we identified may be represented as $\llbracket (\hookrightarrow_x \parallel \hookrightarrow_y); (\llbracket \cdot \rrbracket_x \parallel \llbracket \cdot \rrbracket_y) \rrbracket$.

Series-parallel pomsets with boxes may also be defined by excluded patterns, in the same style as the characterization of series-parallel pomsets [27, 9, 8]. More precisely, one can prove that a pomset $[P]_{\cong}$ is series-parallel iff and only if it does not contain any of the patterns in Figure 4. These may be expressed formally as follows:

$$\mathbf{P}_1 : \exists e_1, e_2, e_3, e_4 \in \mathcal{E}_P : e_1 \leq_P e_3 \wedge e_2 \leq_P e_3 \wedge e_2 \leq_P e_4 \wedge e_1 \not\leq_P e_4 \wedge e_2 \not\leq_P e_1 \wedge e_4 \not\leq_P e_3$$

$$\mathbf{P}_2 : \exists e_1, e_2, e_3 \in \mathcal{E}_P, \exists A, B \in \mathcal{B}_P : e_1 \in A \setminus B \wedge e_2 \in A \cap B \wedge e_3 \in B \setminus A$$

$$\mathbf{P}_3 : \exists e_1, e_2, e_3 \in \mathcal{E}_P, \exists A \in \mathcal{B}_P : e_1 \notin A \wedge e_2, e_3 \in A \wedge e_1 \leq_P e_2 \wedge e_1 \not\leq_P e_3$$

$$\mathbf{P}_4 : \exists e_1, e_2, e_3 \in \mathcal{E}_P, \exists A \in \mathcal{B}_P : e_1 \notin A \wedge e_2, e_3 \in A \wedge e_2 \leq_P e_1 \wedge e_3 \not\leq_P e_1.$$

We omit the proof of this result here, but the interested reader may find it both in the Coq proof and in the online version referenced above.

These four patterns are invariant under isomorphism, since they only use the ordering between events and the membership of events to boxes. This is consistent with SP being a property of pomsets, not posets. This result provides an alternative view of pomsets with boxes: one may see them as *hyper-pomsets*; that is, pomsets in which some events (the boxes) can be labelled with non-empty pomsets (the contents of the boxes). However, it seems that for our purposes the definition we provide is more convenient. In particular, the definition of hyper-pomset homomorphism is more involved.

2.2 Sets of posets

We now lift our operations and relations to sets of posets. This allows us to enrich our syntax with a non-deterministic choice operator.

► **Definition 15** (Orderings on sets of posets). *Let $A, B \subseteq \mathbb{P}_\Sigma$, we define the following:*

Isomorphic inclusion $A \subsetneq B$ iff $\forall P \in A, \exists Q \in B$ such that $P \cong Q$

Isomorphic equivalence $A \cong B$ iff $A \subsetneq B \wedge B \subsetneq A$

Subsumption $A \sqsubseteq B$ iff $\forall P \in A, \exists Q \in B$ such that $P \sqsubseteq Q$.

► **Remark 16.** Isomorphic inclusion and subsumption are partial orders with respect to isomorphic equivalence, which is an equivalence relation.

► **Definition 17** (Operations on sets of posets). *We will use the set-theoretic union of sets of posets, as well as the pointwise liftings of the two products of posets and the boxing operators:*

$$A \otimes B := \{P \otimes Q \mid \langle P, Q \rangle \in A \times B\} \quad [A] := \{[P] \mid P \in A\}$$

$$A \oplus B := \{P \oplus Q \mid \langle P, Q \rangle \in A \times B\}.$$

■ **Table 1** Equational and inequational logic

$\frac{e = f \in A}{A \vdash e = f}$	$A \vdash e = e$	$\frac{A \vdash e = f}{A \vdash f = e}$	$\frac{A \vdash e = f \quad A \vdash f = g}{A \vdash e = g}$
$\sigma, \tau : \Sigma \rightarrow \mathbf{T}_\Sigma, \frac{\forall a \in \Sigma, A \vdash \sigma(a) = \tau(a)}{A \vdash \hat{\sigma}(e) = \hat{\tau}(e)}$			
$\frac{e = f \in A}{A \vdash e \leq f}$	$\frac{f = e \in A}{A \vdash e \leq f}$	$\frac{e \leq f \in A}{A \vdash e \leq f}$	$A \vdash e \leq e$
		$\frac{A \vdash e \leq f \quad A \vdash f \leq g}{A \vdash e \leq g}$	
$\sigma, \tau : \Sigma \rightarrow \mathbf{T}_\Sigma, \frac{\forall a \in \Sigma, A \vdash \sigma(a) \leq \tau(a)}{A \vdash \hat{\sigma}(e) \leq \hat{\tau}(e)}$			

► **Definition 18** (Closure of a set of posets). *The (downwards) closure of a set of posets S is the smallest set containing S that is downwards closed with respect to the subsumption order; that is, $S \downarrow := \{P \in \mathbb{P}_\Sigma \mid \exists Q \in S : P \sqsubseteq Q\}$. Similarly, the upwards closure of S is defined as: $S \uparrow := \{P \in \mathbb{P}_\Sigma \mid \exists Q \in S : P \sqsupseteq Q\}$.*

► **Remark 19.** $(_) \downarrow$ and $(_) \uparrow$ are Kuratowski closure operators [16]; i.e., they satisfy the following properties:

$$\emptyset \downarrow = \emptyset \quad A \subseteq A \downarrow \quad A \downarrow \downarrow = A \downarrow \quad (A \cup B) \downarrow = A \downarrow \cup B \downarrow.$$

(And, similarly, for the upwards closure.) Using downwards-closures, we may express subsumption in terms of isomorphic inclusion:

$$A \sqsubseteq B \quad \Leftrightarrow \quad A \subsetneq B \downarrow \quad \Leftrightarrow \quad A \downarrow \subsetneq B \downarrow.$$

Similarly, the equivalence relation associated with \sqsubseteq , defined as the intersection of the relation and its converse, corresponds to the predicate $A \downarrow \cong B \downarrow$.

► **Definition 20.** *Terms are defined by the following grammar:*

$$e, f \in \mathbf{T}_\Sigma ::= 0 \mid 1 \mid a \mid e; f \mid e \parallel f \mid e + f \mid [e].$$

Terms can be interpreted as finite sets of posets with boxes as follows:

$$\begin{aligned} \llbracket 0 \rrbracket &:= \emptyset & \llbracket 1 \rrbracket &:= \{\mathfrak{c}\} & \llbracket a \rrbracket &:= \{\mathfrak{a}\} \\ \llbracket [e] \rrbracket &:= \llbracket [e] \rrbracket & \llbracket e; f \rrbracket &:= \llbracket e \rrbracket \otimes \llbracket f \rrbracket & \llbracket e + f \rrbracket &:= \llbracket e \rrbracket \cup \llbracket f \rrbracket & \llbracket e \parallel f \rrbracket &:= \llbracket e \rrbracket \oplus \llbracket f \rrbracket. \end{aligned}$$

► **Remark 21.** Interpreted as a program, 0 represents failure: this is a program that aborts the whole execution. +, on the other hand, represents non-deterministic choice. It can be used to model conditional branching.

2.3 Axiomatic presentations of pomset algebra

We now introduce axioms to capture the various order and equivalence relations we introduced over posets and sets of posets. Given a set of axioms A (i.e., universally quantified identities),

■ **Table 2** Axioms

$s;(t;u) = (s;t);u$	(A1)	$e+(f+g) = (e+f)+g$	(C1)
$s \parallel (t \parallel u) = (s \parallel t) \parallel u$	(A2)	$e+f = f+e$	(C2)
$s \parallel t = t \parallel s$	(A3)	$e+e = e$	(C3)
$1; s = s$	(A4)	$0+e = e$	(C4)
$s; 1 = s$	(A5)	$0; e = e; 0 = 0$	(C5)
$1 \parallel s = s$	(A6)	$0 \parallel e = 0$	(C6)
$[[s]] = [s]$	(A7)	$e;(f+g) = (e;f)+(e;g)$	(C7)
$[1] = 1$	(A8)	$(e+f);g = (e;g)+(f;g)$	(C8)
		$e \parallel (f+g) = (e \parallel f)+(e \parallel g)$	(C9)
$(s \parallel t);(u \parallel v) \leq (s;u) \parallel (t;v)$	(B1)	$[0] = 0$	(C10)
$[s] \leq s$	(B2)	$[e+f] = [e] + [f]$	(C11)

we write $A \vdash e = f$ to denote that the pair $\langle e, f \rangle$ belongs to the smallest congruence containing every axiom in A . Equivalently, $A \vdash e = f$ holds iff this statement is derivable in equational logic, as described in Table 1. Similarly, $A \vdash e \leq f$ is the smallest precongruence containing A , where equality axioms are understood as pairs of inequational axioms. An inference system is also provided in Table 1. We will consider the following sets of axioms:

$$\begin{aligned}
 \text{BiMon}_{\square} &:= (\text{A1}) - (\text{A8}) && \text{(Bimonoid with boxes)} \\
 \text{CMon}_{\square} &:= \text{BiMon}_{\square}, (\text{B1}), (\text{B2}) && \text{(Concurrent monoid with boxes)} \\
 \text{SR}_{\square} &:= \text{BiMon}_{\square}, (\text{C1}) - (\text{C11}) && \text{(Bisemiring with boxes)} \\
 \text{CSR}_{\square} &:= \text{SR}_{\square}, (\text{B1}), (\text{B2}). && \text{(Concurrent semiring with boxes)}
 \end{aligned}$$

In the last theory, inequational axioms $e \leq f$ should be read as $e + f = f$. Indeed one can show that for $A \in \{\text{SR}_{\square}, \text{CSR}_{\square}\}$, we have

$$A \vdash e \leq f \Leftrightarrow A \vdash e + f = f \quad A \vdash e = f \Leftrightarrow A \vdash e \leq f \wedge A \vdash f \leq e.$$

These axioms capture the relations \cong and \sqsubseteq :

► **Theorem 22.** *For any pair of terms $s, t \in \text{SP}_{\Sigma}$, the following hold:*

$$[[s]] \cong [[t]] \Leftrightarrow \text{BiMon}_{\square} \vdash s = t \tag{2.1}$$

$$[[s]] \sqsubseteq [[t]] \Leftrightarrow \text{CMon}_{\square} \vdash s \leq t. \tag{2.2}$$

The following lemma allows us to extend seamlessly our completeness theorem from BiMon_{\square} to SR_{\square} and from CMon_{\square} to CSR_{\square} .

► **Lemma 23.** *There exists a function $T_{\square} : \text{T}_{\Sigma} \rightarrow \mathcal{P}_f(\text{SP}_{\Sigma})$ such that: $\text{SR}_{\square} \vdash e = \sum_{s \in T_e} s$ and $[[e]] \cong \{[[s]] \mid s \in T_e\}$.*

From there, we can easily establish the following completeness results:

► **Theorem 24.** *For any pair of terms $e, f \in \text{T}_{\Sigma}$, the following hold:*

$$[[e]] \cong [[f]] \Leftrightarrow \text{SR}_{\square} \vdash e = f \tag{2.3}$$

$$[[e]] \downarrow \cong [[f]] \downarrow \Leftrightarrow \text{CSR}_{\square} \vdash e = f. \tag{2.4}$$

3 Logic for pomsets with boxes

We introduce a logic for reasoning about pomsets with boxes, in the form of a bunched modal logic, in the sense of [21, 7, 5, 1, 23], with substructural connectives corresponding to each of sequential and concurrent composition. Modalities characterize boxes and locality. The logic is also conceptually related to Concurrent Separation Logic [19, 3].

In contrast with other work, pomset logic is a logic of *behaviours*. A behaviour is a run of some program, represented as a pomset. The logic describes such behaviours in terms of the order in which instructions are, or can be, executed, and the separation properties of sub-runs. Note, in particular, that we do not define any notion of *state*. On the contrary, existing approaches, such as dynamic logic and Hennessy–Milner logic for example, put the emphasis on the state of the machine before and after running the program. Typically, the assertion language describes the memory-states, and some accessibility relations between them. The semantics then relies on labelled transition systems to interpret action modalities.

Here, the satisfaction relation (given in Definition 26) directly defines a relation between sets of behaviours and formulas. An intuitionistic version of the semantics given in Definition 26 might be set up — cf. Tarski’s semantics and the semantics of relevant logic — in terms of (ternary) relations on behaviours.

3.1 Pomset logic: definitions

We generate the set of formulas F_Σ and the set of positive formulas F_Σ^+ as follows:

$$\begin{aligned} \phi, \psi \in F_\Sigma^+ &::= \perp \mid a \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \blacktriangleright \psi \mid \phi \star \psi \mid [\phi] \mid \langle \phi \rangle \\ \phi, \psi \in F_\Sigma &::= \perp \mid a \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \blacktriangleright \psi \mid \phi \star \psi \mid [\phi] \mid \langle \phi \rangle \mid \neg \phi \end{aligned}$$

► **Remark 25.** Here the atomic predicates are chosen to be exactly Σ . Another natural choice would be a separate set *Prop* of atomic predicates, together with a valuation $v : Prop \rightarrow \mathcal{P}(\Sigma)$ to indicate which actions satisfy which predicate. Both definitions are equivalent:

- to encode a formula over *Prop* as a formula over Σ , simply replace every predicate $p \in Prop$ with the formula $\bigvee_{a \in v(p)} a$
- to encode a formula over Σ as one over *Prop*, we need to make the customary assumption that $\forall a \in \Sigma, \exists p \in Prop : v(p) = \{a\}$.

These formulas are interpreted over posets. We define a satisfaction relation \models_R that is parametrized by a relation $R \subseteq \mathcal{P}_\Sigma \times \mathcal{P}_\Sigma$ (to be instantiated later on with \cong , \sqsubseteq , and \sqsupseteq).

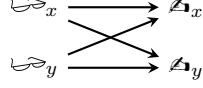
► **Definition 26.** $P \models_R \phi$ is defined by induction on $\phi \in F_\Sigma$:

- $P \models_R \perp$ iff $R(P, \epsilon)$
- $P \models_R a$ iff $R(P, a)$
- $P \models_R \neg \phi$ iff $P \not\models_R \phi$
- $P \models_R \phi \vee \psi$ iff $P \models_R \phi$ or $P \models_R \psi$
- $P \models_R \phi \wedge \psi$ iff $P \models_R \phi$ and $P \models_R \psi$
- $P \models_R \phi \blacktriangleright \psi$ iff $\exists P_1, P_2$ such that $R(P, P_1 \otimes P_2)$ and $P_1 \models_R \phi$ and $P_2 \models_R \psi$
- $P \models_R \phi \star \psi$ iff $\exists P_1, P_2$ such that $R(P, P_1 \oplus P_2)$ and $P_1 \models_R \phi$ and $P_2 \models_R \psi$
- $P \models_R [\phi]$ iff $\exists Q$ such that $R(P, [Q])$ and $Q \models_R \phi$
- $P \models_R \langle \phi \rangle$ iff $\exists P', Q$ such that $R(P, P')$ and $P' \ni Q$ and $Q \models_R \phi$.

The operator $[-]$ describes the (encapsulated) properties of boxed terms. The operator $\langle - \rangle$ identifies a property of a term that is obtained by removing parts, including boxes and events,

of its satisfying term (i.e., its world) such that remainder satisfies the formula that it guards. The meanings of these operators are discussed more fully in Section 4.2. Note that \models_{\sqsupseteq} and \models_{\sqsubseteq} will only be used with positive formulas. Given a formula ϕ and a relation R , we may define the R -semantics of ϕ as $\llbracket \phi \rrbracket_R := \{P \in \mathbb{P}_\Sigma \mid P \models_R \phi\}$.

► **Example 27.** Recall the problematic pattern we saw in the running example; i.e.,



This pattern can be represented by the formula **conflict** := $((\varpi_x \star \varpi_y) \blacktriangleright (\mathcal{A}_x \star \mathcal{A}_y))$.

We may also interpret these formulas over sets of posets. We consider here two ways a set of posets X may satisfy a formula:

- X satisfies ϕ *universally* if every poset in X satisfies ϕ ;
- X satisfies ϕ *existentially* if some poset in X satisfies ϕ .

Combined with our three satisfaction relations for pomsets, this yields six definitions:

$$\begin{array}{ll} X \models_{\cong}^{\forall} \phi \text{ iff } \forall P \in X, P \models_{\cong} \phi & X \models_{\cong}^{\exists} \phi \text{ iff } \exists P \in X, P \models_{\cong} \phi \\ X \models_{\sqsupseteq}^{\forall} \phi \text{ iff } \forall P \in X, P \models_{\sqsupseteq} \phi & X \models_{\sqsupseteq}^{\exists} \phi \text{ iff } \exists P \in X, P \models_{\sqsupseteq} \phi \\ X \models_{\sqsubseteq}^{\forall} \phi \text{ iff } \forall P \in X, P \models_{\sqsubseteq} \phi & X \models_{\sqsubseteq}^{\exists} \phi \text{ iff } \exists P \in X, P \models_{\sqsubseteq} \phi. \end{array}$$

For a term $e \in \mathbb{T}_\Sigma$, we write $e \models_R^y \phi$ to mean $\llbracket e \rrbracket \models_R^y \phi$. In terms of R -semantics, these definitions may be formalized as:

$$e \models_R^{\exists} \phi \Leftrightarrow \llbracket e \rrbracket \cap \llbracket \phi \rrbracket_R \neq \emptyset \qquad e \models_R^{\forall} \phi \Leftrightarrow \llbracket e \rrbracket \subseteq \llbracket \phi \rrbracket_R. \quad (3.1)$$

3.2 Properties of pomset logic

We now discuss some of the properties of pomset logic. First, notice that if the relation R is transitive, then for any posets P, Q and any formula $\phi \in \mathbf{F}_\Sigma^+$, we have that:

$$P R Q \text{ and } Q \models_R \phi \Rightarrow P \models_R \phi. \quad (3.2)$$

If, additionally, R is symmetric, this property may be strengthened to

$$\forall P, Q \in \mathbb{P}_\Sigma, \forall \phi \in \mathbf{F}_\Sigma, \text{ if } P R Q, \text{ then } P \models_R \phi \Leftrightarrow Q \models_R \phi. \quad (3.3)$$

Furthermore, increasing the relation R increases the satisfaction relation as well:

$$R \subseteq R' \Rightarrow \forall \phi \in \mathbf{F}_\Sigma^+, \forall P \in \mathbb{P}_\Sigma, P \models_R \phi \Rightarrow P \models_{R'} \phi. \quad (3.4)$$

From these observations and (3.1), we obtain the following characterizations of the universal satisfaction relations for $R \in \{\cong, \sqsubseteq, \sqsupseteq\}$:

$$e \models_{\cong}^{\forall} \phi \Leftrightarrow \llbracket e \rrbracket \subsetneq \llbracket \phi \rrbracket_{\cong} \quad (3.5)$$

$$e \models_{\sqsubseteq}^{\forall} \phi \Leftrightarrow \llbracket e \rrbracket \sqsubseteq \llbracket \phi \rrbracket_{\sqsubseteq} \quad (3.6)$$

$$e \models_{\sqsupseteq}^{\forall} \phi \Leftrightarrow \forall P \in \llbracket e \rrbracket, \exists Q \in \llbracket \phi \rrbracket_{\sqsupseteq} : P \sqsupseteq Q. \quad (3.7)$$

Additionally, the following preservation properties hold for sets of posets:

$$e \subsetneq f \Rightarrow \forall \phi \in \mathbf{F}_\Sigma, (e \models_{\cong}^{\exists} \phi \Rightarrow f \models_{\cong}^{\exists} \phi) \wedge (f \models_{\cong}^{\forall} \phi \Rightarrow e \models_{\cong}^{\forall} \phi) \quad (3.8)$$

$$e \sqsubseteq f \Rightarrow \forall \phi \in \mathbf{F}_\Sigma^+, (e \models_{\sqsubseteq}^{\exists} \phi \Rightarrow f \models_{\sqsubseteq}^{\exists} \phi) \wedge (f \models_{\sqsubseteq}^{\forall} \phi \Rightarrow e \models_{\sqsubseteq}^{\forall} \phi). \quad (3.9)$$

We can build formulas from series-parallel terms: $\phi(a) := a$, $\phi(1) := \perp$, $\phi([s]) := [\phi(s)]$, $\phi(s; t) := \phi(s) \blacktriangleright \phi(t)$, and $\phi(s \parallel t) := \phi(s) \star \phi(t)$. Using T_- , we generalize this construction our full syntax: given a term $e \in \mathbf{T}_\Sigma$, we define the formula $\Phi(e) := \bigvee_{s \in T_e} \phi(s)$. These formulas are closely related to terms thanks to the following lemma:

► **Lemma 28.** *For any term $s \in \mathbf{SP}_\Sigma$ and any poset P , we have:*

$$P \models_{\cong} \phi(s) \Leftrightarrow P \cong \llbracket s \rrbracket \quad P \models_{\supseteq} \phi(s) \Leftrightarrow P \supseteq \llbracket s \rrbracket \quad P \models_{\sqsubseteq} \phi(s) \Leftrightarrow P \sqsubseteq \llbracket s \rrbracket.$$

For a term $e \in \mathbf{T}_\Sigma$ and a set of posets $X \subseteq \mathbf{P}_\Sigma$, we have:

$$X \models_{\forall}^{\cong} \Phi(e) \Leftrightarrow X \sqsubset \llbracket e \rrbracket \quad X \models_{\sqsubseteq}^{\forall} \Phi(e) \Leftrightarrow X \sqsubseteq \llbracket e \rrbracket.$$

As an immediate corollary, for any $e \in \mathbf{T}_\Sigma$ and any $s \in \mathbf{SP}_\Sigma$, we obtain that:

$$e \models_{\cong}^{\exists} \phi(s) \Leftrightarrow \llbracket s \rrbracket \in \llbracket e \rrbracket \quad e \models_{\supseteq}^{\exists} \phi(s) \Leftrightarrow \llbracket s \rrbracket \in \llbracket e \rrbracket \downarrow \quad (3.10)$$

We can now establish adequacy lemmas. These should be understood as appropriate formulations of the completeness theorems relating operational equivalence and logical equivalence in the sense of van Benthem [2] and Hennessy–Milner [10, 18] for this logic (cf. [1]). From the results we have established so far, we may directly prove the following:

► **Proposition 29.** *For a pair of series-parallel terms $s, t \in \mathbf{SP}_\Sigma$,*

$$\text{BiMon}_\square \vdash s = t \Leftrightarrow \forall \phi \in \mathbf{F}_\Sigma, (\llbracket s \rrbracket \models_{\cong} \phi \Leftrightarrow \llbracket t \rrbracket \models_{\cong} \phi) \quad (3.11)$$

$$\text{CMon}_\square \vdash s \leq t \Leftrightarrow \forall \phi \in \mathbf{F}_\Sigma^+, (\llbracket s \rrbracket \models_{\supseteq} \phi \Rightarrow \llbracket t \rrbracket \models_{\supseteq} \phi). \quad (3.12)$$

This extends to sets of pomsets in the following sense:

► **Proposition 30.** *Given two terms $e, f \in \mathbf{T}_\Sigma$, the following equivalences hold:*

$$\text{SR}_\square \vdash e \leq f \Leftrightarrow (\forall \phi, e \models_{\supseteq}^{\exists} \phi \Rightarrow f \models_{\supseteq}^{\exists} \phi) \Leftrightarrow (\forall \phi, f \models_{\cong}^{\forall} \phi \Rightarrow e \models_{\cong}^{\forall} \phi) \quad (3.13)$$

$$\text{SR}_\square \vdash e = f \Leftrightarrow (\forall \phi, e \models_{\supseteq}^{\exists} \phi \Leftrightarrow f \models_{\supseteq}^{\exists} \phi) \Leftrightarrow (\forall \phi, e \models_{\cong}^{\forall} \phi \Leftrightarrow f \models_{\cong}^{\forall} \phi) \quad (3.14)$$

$$\text{CSR}_\square \vdash e \leq f \Leftrightarrow (\forall \phi, e \models_{\supseteq}^{\exists} \phi \Rightarrow f \models_{\supseteq}^{\exists} \phi) \Leftrightarrow (\forall \phi, f \models_{\sqsubseteq}^{\forall} \phi \Rightarrow e \models_{\sqsubseteq}^{\forall} \phi) \quad (3.15)$$

$$\text{CSR}_\square \vdash e = f \Leftrightarrow (\forall \phi, e \models_{\supseteq}^{\exists} \phi \Leftrightarrow f \models_{\supseteq}^{\exists} \phi) \Leftrightarrow (\forall \phi, e \models_{\sqsubseteq}^{\forall} \phi \Leftrightarrow f \models_{\sqsubseteq}^{\forall} \phi). \quad (3.16)$$

4 Local Reasoning

Some of the discussions in this section do not rely on which satisfaction relation we pick. When this is the case, we use the symbol \models to mean any of the relations $\models_{\cong}, \models_{\supseteq}, \models_{\sqsubseteq}$.

4.1 Modularity

Pomset logic enjoys a high level of compositionality, much like algebraic logic. Formally, this comes from the following principle:

$$\text{If } e \models \phi \text{ and } \forall a, \sigma a \models \tau a, \text{ then } \hat{\sigma} e \models \hat{\tau} \phi.$$

This makes possible the following verification scenario: Let P be a large program, involving a number of simpler sub-programs P_1, \dots, P_n . We may simplify P by replacing the sub-programs by uninterpreted symbols x_1, \dots, x_n . We then check that this simplified program satisfies a formula Φ , the statement of which might involve the x_i . We then separately determine for each sub-program P_i some specification ϕ_i . Finally, using the principle we just stated, we can show that the full program P satisfies the formula Φ' , obtained by replacing the x_i with ϕ_i .

4.2 Frame rule

A key objective of applied, modelling-oriented, work in logic and semantics is to understand systems — such as complex programs, large-scale distributed systems, and organizations — compositionally. That is, we seek to understand how the system is made of components that can be understood independently of one another. A key aspect of this is what has become known as *local reasoning*. That is, that the pertinent logical properties of the components of a system should be independent of their context.

In the world of Separation Logic [26, 12, 19], for reasoning about how computer programs manipulate memory, O’Hearn, Reynolds, and Yang [22] suggest that

‘To understand how a program works, it should be possible for reasoning and specification to be confined to the cells that the program actually accesses. The value of any other cell will automatically remain unchanged.’

In this context, a key idea is that of the ‘footprint’ of a program; that is, that part of memory that is, in an appropriate sense, used by the program [24]. If, in an appropriate sense, a program executes correctly, or ‘safely’, on its footprint, then the so-called ‘frame property’ ensures that the resources present outside of the footprint and, by implication, their inherent logical properties, are unchanged by the program.

In the setting of Separation Logic, the frame property is usually represented by a Hoare-triple rule of the form

$$\frac{\{\phi\}C\{\psi\}}{\{\phi * \chi\}C\{\psi * \chi\}} \quad C \text{ is independent of } \chi.$$

That is, the formula χ does not include any variables (from the memory) that are modified by the program C .

In order to formulate the frame property in our framework, we first fix the notion of independence between a program and a formula. We say that a pomset P is *R-independent* of a formula ϕ , written $P \#_R \phi$ if $P \not\models_R \llbracket \phi \rrbracket$. Since independence is meant to prevent overlap, the use of the $\llbracket - \rrbracket$ modality should come as no surprise.

To explain the need for the $\llbracket - \rrbracket$ modality, first consider a pomset P satisfying $\llbracket \phi \rrbracket$. To extract a witness of this fact, we must remove parts of P , including boxes and events, such that the remainder satisfies ϕ . However, there are no restrictions on the relationship between the remaining events and those we have deleted. In a sequence of three events, we are allowed to keep the two extremities, and delete the middle one. In contrast, to get a witness of $\llbracket \phi \rrbracket$, we need to identify a box on P whose contents satisfy ϕ , and remove all events external to that box. The result is that the deleted events, that is, the context of our witness, can only appear outside the box, and must treat all events inside uniformly. In other words, these events can interact with the behaviour encapsulated in the box, but cannot interact with individual components inside. For this reason, the frame properties given in Proposition 31 are expressed using $\llbracket \phi \rrbracket$ — that is, the encapsulation of ϕ — rather than ϕ .

With this definition, we can now state three frame rules, enabling local reasoning with respect to the parallel product, sequential prefixing, and sequential suffixing.

► **Proposition 31** (Frame properties). *If $P \#_{\cong} \phi$, and $Q \models_{\cong} \llbracket \phi \rrbracket$, then it holds that:*

- (i) $\forall \psi \in \mathbf{F}_{\Sigma}, P \models_{\cong} \psi \Leftrightarrow P \oplus Q \models_{\cong} \psi * \llbracket \phi \rrbracket$;
- (ii) $\forall \psi \in \mathbf{F}_{\Sigma}, P \models_{\cong} \psi \Leftrightarrow P \otimes Q \models_{\cong} \psi \blacktriangleright \llbracket \phi \rrbracket$;
- (iii) $\forall \psi \in \mathbf{F}_{\Sigma}, P \models_{\cong} \psi \Leftrightarrow Q \otimes P \models_{\cong} \llbracket \phi \rrbracket \blacktriangleright \psi$.

■ **Figure 5** Voting protocol

$$\begin{aligned}
\text{VoteProc} &:= \text{Choose}; \text{Publish} \\
\text{Choose} &:= [\text{Vote}(1)] \parallel \cdots \parallel [\text{Vote}(n)] \\
\text{Vote}(i) &:= \sum_{1 \leq j \leq k} \boxtimes_{i,j}; \hookrightarrow_j; \uparrow; \triangleleft_j \\
\text{Publish} &:= \boxtimes_1 \parallel \cdots \parallel \boxtimes_n \\
\boxtimes_i &: \text{the contents of counters } c_1, \dots, c_k \text{ is sent to voter } v_i \\
\boxtimes_{i,j} &: \text{voter } v_i \text{ chooses counter } c_j \\
\hookrightarrow_j &: \text{the content of counter } c_j \text{ is loaded into a local variable} \\
\uparrow &: \text{the local variable is incremented} \\
\triangleleft_j &: \text{the content of the local variable is stored in counter } c_j
\end{aligned}$$

► **Remark 32.** Note that this lemma does not hold for \sqsubseteq or \sqsupseteq instead of \simeq . The left-to-right implications always hold, but the converse may fail. However, this principle may be extended to sets of pomsets. Indeed, if we define the independence relation for sets of pomsets as $A \#_R \phi := \forall P \in A, P \#_R \phi$, we may prove that Proposition 31 holds for both $\models_{\sqsubseteq}^{\forall}$ and $\models_{\sqsupseteq}^{\exists}$.

4.3 Example

In this section, we present an example program, and showcase reasoning principles of pomset logic. In particular, we will highlight the use of local reasoning when appropriate.

Consider the following voting protocol: a fixed number of voters, v_1, \dots, v_n , are each asked to increment one of the counters c_1, \dots, c_k . The tally is then sent to each of the v_i , to inform them of the result. The increment is implemented similarly to our running example of the distributed counter (Example 1). The implementation of the protocol is displayed in Figure 5, together with the intended semantics of the atomic actions.

Conflict

As in Example 1, if we forgo the boxes in **Choose**, we cannot enforce mutual exclusion. Recall that the undesirable behaviour is captured by the following formula:

$$\mathbf{conflict}_j := ((\hookrightarrow_j \star \hookrightarrow_j) \blacktriangleright (\triangleleft_j \star \triangleleft_j))$$

We may see this by defining an alternative (faulty) protocol:

$$\text{VoteProc}' := (\text{Vote}(1) \parallel \cdots \parallel \text{Vote}(n)); \text{Publish}$$

and then checking that this protocol displays the behaviour we wanted to avoid:

$$\text{VoteProc}' \models_{\sqsupseteq}^{\exists} \mathbf{conflict}_j.$$

This statement should be read as ‘there is a pomset in $\llbracket \text{VoteProc}' \rrbracket$ that is larger than one containing a conflict’. We can show the existence of this ‘bug’ by local reasoning. We may

first prove that $\text{Vote}(i) \parallel \text{Vote}(i') \models_{\exists}^{\exists} \mathbf{conflict}_j$ (for some arbitrary $i \neq i'$). The properties of $(-)$ then allow us to deduce that

$$\text{VoteProc}' \cong (\text{Vote}(i) \parallel \text{Vote}(i') \parallel \dots); \dots \models_{\exists}^{\exists} ((\mathbf{conflict}_j)) \equiv \mathbf{conflict}_j.$$

The implementation in Figure 5 avoids this problem, and indeed it holds that:

$$\text{VoteProc} \not\models_{\exists}^{\exists} \mathbf{conflict}_j.$$

However, showing that this formula is *not* satisfied by the program is less straightforward and, in particular, cannot be done locally: we have to enumerate all possible sub-pomsets, and check that none provide a suitable witness.

Sequential separation

In our protocol, the results of the vote are only communicated after every participant has voted. This is specified by the following statement:

$$\mathbf{SendAfterVote} := \left(\neg \left(\bigvee_i \mathfrak{a}_i \right) \right) \blacktriangleright \left(\neg \left(\bigvee_{i,j} \boxtimes_{i,j} \right) \right).$$

This may be checked modularly. Indeed, one may prove by simple syntactic analysis that

$$\mathbf{Choose} \models_{\cong}^{\forall} \left(\neg \left(\bigvee_i \mathfrak{a}_i \right) \right) \quad \text{and} \quad \mathbf{Publish} \models_{\cong}^{\forall} \left(\neg \left(\bigvee_{i,j} \boxtimes_{i,j} \right) \right).$$

Therefore, we may combine these to get that:

$$\text{VoteProc} = \mathbf{Choose}; \mathbf{Publish} \models_{\cong}^{\forall} \left(\neg \left(\bigvee_i \mathfrak{a}_i \right) \right) \blacktriangleright \left(\neg \left(\bigvee_{i,j} \boxtimes_{i,j} \right) \right) = \mathbf{SendAfterVote}.$$

For voter i , two of the most meaningful steps are $\boxtimes_{i,j}$ and \mathfrak{a}_i , i.e. when the vote is cast and when the result of the vote is forwarded to them. Using the macro $\mathbf{choose}_i := \bigvee_j \boxtimes_{i,j}$, we can specify that during the protocol, each voter first votes, and then gets send the result:

$$\mathbf{VoteThenSend} := ((\mathbf{choose}_1 \blacktriangleright \mathfrak{a}_1) \star \dots \star (\mathbf{choose}_n \blacktriangleright \mathfrak{a}_n)).$$

Unique votes

Another important feature of this protocol is that each voter may only cast a single vote. Knowing that each voter controls a single box, we express this property with the statement:

$$\text{VoteProc} \models_{\exists}^{\exists} \bigvee_{j,j'} ((\mathfrak{a}_j \star \mathfrak{a}_{j'})).$$

Since we use the relation $\models_{\exists}^{\exists}$ with the connective \star , we allow any possible ordering of the two write events. The only constraint is that there should be at least two of them in the same box. As for the ‘conflict’ property, if the ‘bad’ behaviour were to happen, one could prove it compositionally. However, disproving the existence of such a behaviour is a more global process, involving the exploration of all possible sub-pomsets.

Frame property

As we have seen in previous examples, proving that a formula does not hold can be challenging, because the non-existence of a local pattern is *not* a local property. We may circumvent this problem by adding more boxes in both programs and formulas. This is related to a common pattern in parallel programming: in a multi-threaded program, one may insert fences to ‘tame’ concurrency. Doing so simplifies program analysis, at the cost of some efficiency. Similarly, since adding boxes restricts behaviours — thus disallowing some possible optimizations — the analysis of a program becomes simpler and more efficient.

We illustrate this with the following statement:

$$[\text{Choose}]; [\text{Publish}] \not\models_{\cong}^{\forall} (\llbracket \text{A} \blacktriangleright \text{A} \rrbracket) \blacktriangleright [\phi] \quad \text{where } \phi := \neg(\perp \vee (\llbracket \text{A} \rrbracket)) .$$

$\llbracket \text{A} \blacktriangleright \text{A} \rrbracket$ indicates that two ‘write’ instructions can be executed in sequence, while ϕ denotes a non-empty pomset, not containing any boxes with a ‘write’ event inside. We can first prove properties of the subprograms:

$$[\text{Publish}] \models_{\cong}^{\forall} [\phi] \quad [\text{Choose}] \not\models_{\cong}^{\forall} (\llbracket \phi \rrbracket) \quad [\text{Choose}] \not\models_{\cong}^{\forall} (\llbracket \text{A} \blacktriangleright \text{A} \rrbracket) .$$

Since $[\text{Choose}] \#_{\cong} \phi$ and $[\text{Publish}] \models_{\cong}^{\forall} [\phi]$, we obtain from the frame rule that

$$[\text{Choose}]; [\text{Publish}] \models_{\cong}^{\forall} (\llbracket \text{A} \blacktriangleright \text{A} \rrbracket) \blacktriangleright [\phi] \Leftrightarrow [\text{Choose}] \models_{\cong}^{\forall} (\llbracket \text{A} \blacktriangleright \text{A} \rrbracket) .$$

Since we have locally disproved the later, we may deduce that the former does not hold.

5 Future work

In this paper, we have not considered the CKA operator $-^*$. A natural further step would be to do so, with the corresponding need to consider versions of pomset logic with fixed points. Connections with Hoare-style program logics, such as Concurrent Separation Logic [3, 20] with its concrete semantics, should also be considered.

Our satisfaction relation over pomsets is defined inductively. However, the satisfaction relations we define for sets of pomsets is not: we define in terms of the former relation. For practical purposes, such as model-checking, it would be useful to have a similar inductive definition for sets of pomsets.

It is also worth noticing that the definitions and statements in Section 2 are straightforward generalizations of their counterparts in CKA (without boxes); even the proofs of those results follow a similar strategy. However, we could reuse almost no result from CKA: instead we had to reprove everything from scratch. This situation is deeply unsatisfactory, and we plan on investigating techniques to better ‘recycle’ proofs in this context. Recent work on (C)KA with *hypotheses* [6, 14] seems to be a step towards this goal.

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A

 Omitted proofs

A.1 Posets up to isomorphisms: the free bimonoid

In this section, we prove the following soundness and completeness theorem:

$$\llbracket s \rrbracket \cong \llbracket t \rrbracket \Leftrightarrow \text{BiMon}_{\square} \vdash s = t. \quad (2.1)$$

The key ingredient in this proof is the construction of a syntactic version of the restriction operator, which extracts a subpomsets out of a pomsets, guided by a subset of events.

► **Definition 33** (Syntactic restriction). *Let $s \in \text{SP}_{\Sigma}$ be a term and $A \subseteq \mathcal{E}_{\llbracket s \rrbracket}$ a set of events. The syntactic restriction of s to A , written $\pi_A(s)$ is defined by induction on terms as follows:*

$$\begin{aligned} \pi_A(1) &:= 1 & \pi_A(s; t) &:= \pi_{A \cap \mathcal{E}_{\llbracket s \rrbracket}}(s); \pi_{A \cap \mathcal{E}_{\llbracket t \rrbracket}}(t) \\ \pi_A(a) &:= \begin{cases} 1 & \text{if } A = \emptyset \\ a & \text{otherwise} \end{cases} & \pi_A(s \parallel t) &:= \pi_{A \cap \mathcal{E}_{\llbracket s \rrbracket}}(s) \parallel \pi_{A \cap \mathcal{E}_{\llbracket t \rrbracket}}(t). \\ \pi_A(\llbracket s \rrbracket) &:= \begin{cases} \llbracket s \rrbracket & \text{if } A = \mathcal{E}_{\llbracket e \rrbracket} \\ \pi_A(s) & \text{otherwise} \end{cases} \end{aligned}$$

The main properties of this operator are stated in the following lemma:

► **Lemma 34.** *For any $s \in \text{SP}_{\Sigma}$ and $A \subseteq \mathcal{E}_{\llbracket s \rrbracket}$ the following holds:*

- (i) $\llbracket \pi_A(s) \rrbracket \cong \llbracket s \rrbracket \downarrow_A$;
- (ii) *if A is prefix and nested, then $\text{BiMon}_{\square} \vdash \pi_A(s); \pi_{\overline{A}}(s) = s$;*
- (iii) *if A is isolated and nested, then $\text{BiMon}_{\square} \vdash \pi_A(s) \parallel \pi_{\overline{A}}(s) = s$.*

Before getting on with the proof of the theorem, we establish the following statements:

$$\mathcal{E}_{\llbracket t \rrbracket} = \emptyset \Rightarrow \text{BiMon}_{\square} \vdash t = 1 \quad (A.1)$$

$$\mathcal{E}_{\llbracket s \rrbracket} \in \mathcal{B}_{\llbracket s \rrbracket} \Rightarrow \exists t : \text{BiMon}_{\square} \vdash s = \llbracket t \rrbracket \wedge \mathcal{E}_{\llbracket t \rrbracket} \notin \mathcal{B}_{\llbracket t \rrbracket} \quad (A.2)$$

Proof of statement (A.1). Since $\mathcal{E}_{\llbracket t \rrbracket} = \emptyset$, t does not feature any symbol from Σ , meaning $t \in \text{SP}_{\emptyset}$. We may then show that for every term $t \in \text{SP}_{\emptyset}$ we have $\text{BiMon}_{\square} \vdash t = 1$. ◀

Proof of statement (A.2). We proceed by induction on s :

$s = 1$, $s = a$ contradicts the premiss;

$s = \llbracket s' \rrbracket$ we have to consider two cases:

- $\mathcal{E}_{\llbracket s' \rrbracket} \in \mathcal{B}_{\llbracket s' \rrbracket}$: in this case, by induction we get t such that $\mathcal{E}_{\llbracket t \rrbracket} \notin \mathcal{B}_{\llbracket t \rrbracket}$, and $\text{BiMon}_{\square} \vdash s = \llbracket s' \rrbracket = \llbracket \llbracket t \rrbracket \rrbracket = \llbracket t \rrbracket$.
- $\mathcal{E}_{\llbracket s' \rrbracket} \notin \mathcal{B}_{\llbracket s' \rrbracket}$: pick $t = s'$.

$s = s_1; s_2$ we know the following facts:

$$\begin{aligned} \mathcal{E}_{\llbracket s_1 \rrbracket} \cup \mathcal{E}_{\llbracket s_2 \rrbracket} &= \mathcal{E}_{\llbracket s \rrbracket} \in \mathcal{B}_{\llbracket s \rrbracket} = \mathcal{B}_{\llbracket s_1 \rrbracket} \cup \mathcal{B}_{\llbracket s_2 \rrbracket} \\ \forall \beta \in \mathcal{B}_{\llbracket s_i \rrbracket}, \beta &\subseteq \mathcal{E}_{\llbracket s_i \rrbracket} \\ \mathcal{E}_{\llbracket s_1 \rrbracket} \cap \mathcal{E}_{\llbracket s_2 \rrbracket} &= \emptyset \end{aligned}$$

From them we deduce that either $\mathcal{E}_{\llbracket s_1 \rrbracket} = \emptyset$ or $\mathcal{E}_{\llbracket s_2 \rrbracket} = \emptyset$. We conclude by applying the induction hypothesis to the appropriate subterm, and use (A.1) to conclude.

$s = s_1 \parallel s_2$ same as $s_1; s_2$. ◀

We may now prove (2.1):

Proof of (2.1). As often for this kind of result, the right-to-left implication, i.e., soundness, is very simple to check, by a simple induction on the derivation.

For the converse direction, we prove the following statement by induction on the term s :

$$\forall t \in \mathbf{SP}_\Sigma, \llbracket s \rrbracket \cong \llbracket t \rrbracket \Rightarrow \mathbf{BiMon}_\square \vdash s = t.$$

$s = 1$: follows from (A.1).

$s = a$: we prove the result by induction on t :

$t = 1, t = [t']$: impossible since $\llbracket t \rrbracket \cong \mathfrak{o}$;

$t = b$: since $\llbracket t \rrbracket \cong \mathfrak{o}$ this means $a = b$; i.e., by reflexivity $\mathbf{BiMon}_\square \vdash t = a$;

$t = t_1 ; t_2$: since \mathfrak{o} has a single event, and $\mathcal{E}_{\llbracket t_1 ; t_2 \rrbracket} = \mathcal{E}_{\llbracket t_1 \rrbracket} \uplus \mathcal{E}_{\llbracket t_2 \rrbracket}$, that event must be either in $\mathcal{E}_{\llbracket t_1 \rrbracket}$ or in $\mathcal{E}_{\llbracket t_2 \rrbracket}$, and the other term has no event. The term that has no event, by (A.1), is provably equal to 1. The term containing an event is isomorphic to \mathfrak{o} , so by induction it is provably equal to a . Hence we get that t is provably equal to either $a ; 1$ or $1 ; a$, both of which are provably equal to a .

$t = t_1 \parallel t_2$: same as $t_1 ; t_2$.

$s = s_1 ; s_2$: let $A := \mathcal{E}_{\llbracket s_1 \rrbracket} \subseteq \mathcal{E}_{\llbracket s \rrbracket}$. Notice that $\overline{A} = \mathcal{E}_{\llbracket s \rrbracket} \setminus \mathcal{E}_{\llbracket s_1 \rrbracket} = \mathcal{E}_{\llbracket s_2 \rrbracket}$. Let ϕ be the isomorphism from $\llbracket s \rrbracket$ to $\llbracket t \rrbracket$ and let $t_1 := \pi_{\phi(A)}(t)$ and $t_2 := \pi_{\overline{\phi(A)}}(t)$. Since A is nested and prefix, so is its image by the isomorphism ϕ . Therefore by Lemma 34 we get that: $\mathbf{BiMon}_\square \vdash t = t_1 ; t_2$. By properties of isomorphisms, we can also see that for any set $X \subseteq \mathcal{E}_{\llbracket s \rrbracket}$ we have $\llbracket s \rrbracket \downarrow_X \cong \llbracket t \rrbracket \downarrow_{\phi(X)}$. Hence we have $\llbracket t_1 \rrbracket \cong \llbracket t \rrbracket \downarrow_{\phi(A)} \cong \llbracket s \rrbracket \downarrow_A \cong \llbracket s_1 \rrbracket$. Similarly, and because by bijectivity of ϕ we have $\overline{\phi(A)} = \phi(\overline{A})$, we get $\llbracket t_2 \rrbracket \cong \llbracket s_2 \rrbracket$. We may thus conclude by induction that $\mathbf{BiMon}_\square \vdash s_i = t_i$ ($i \in \{1, 2\}$); i.e.,

$$\mathbf{BiMon}_\square \vdash s = s_1 ; s_2 = t_1 ; t_2 = t.$$

$s = s_1 \parallel s_2$: same as $s_1 ; s_2$. ◀

A.2 Posets up to subsumption: the free concurrent monoid

In this section we show how to prove the following:

$$\llbracket s \rrbracket \sqsubseteq \llbracket t \rrbracket \Leftrightarrow \mathbf{CMon}_\square \vdash s \leq t \tag{2.2}$$

To prove this result, we will deal with the two extra axioms (B1) and (B2) separately. In order to do so, we define the following sub-orders of \sqsubseteq :

- we write $P \sqsubseteq_b Q$ if there is an order-reflecting poset homomorphism $\phi : Q \rightarrow P$;
- we write $P \sqsubseteq_o Q$ if there is an box-reflecting poset homomorphism $\phi : Q \rightarrow P$.

► **Lemma 35** (Factorization of subsumption). *If $P \sqsubseteq Q$, then there are R_1, R_2 such that: $P \sqsubseteq_o R_1 \sqsubseteq_b Q$ and $P \sqsubseteq_b R_2 \sqsubseteq_o Q$. In other words, $\sqsubseteq = \sqsubseteq_o \circ \sqsubseteq_b$ and $\sqsubseteq = \sqsubseteq_b \circ \sqsubseteq_o$.*

Proof. Let $\phi : Q \rightarrow P$ be a poset homomorphism witnessing $P \sqsubseteq Q$. We may define $R_1 := \langle \mathcal{E}_Q, \leq_Q, \lambda_Q, \{B \mid \phi(B) \in \mathcal{B}_P\} \rangle$ and $R_2 := \langle \mathcal{E}_P, \leq_P, \lambda_P, \phi(\mathcal{B}_Q) \rangle$. Checking that $P \sqsubseteq_o R_1 \sqsubseteq_b Q$ and $P \sqsubseteq_b R_2 \sqsubseteq_o Q$ hold is a simple matter of unfolding definitions. ◀

We also notice the following properties of \sqsubseteq_b and \sqsubseteq_o with respect to the forbidden patterns of series-parallel posets:

- **Lemma 36.** *Let P, Q be two posets:*
- (i) *if $P \sqsubseteq_b Q$, then P contains P_1 iff Q contains P_1 ;*
 - (ii) *if $P \sqsubseteq_o Q$, then P contains P_2 iff Q contains P_2 ;*

- (iii) if $P \sqsubseteq_b Q$ and Q contains \mathbf{P}_3 , then P contains \mathbf{P}_3 ;
- (iv) if $P \sqsubseteq_b Q$ and Q contains \mathbf{P}_4 , then P contains \mathbf{P}_4 .

We now prove the first half of the theorem:

► **Lemma 37.** *If $\llbracket s \rrbracket \sqsubseteq_b \llbracket t \rrbracket$, then $\text{CMon}_\square \vdash s \leq t$.*

Proof. First, we define the following operator $\mathcal{H}(_) : \mathbb{P}_\Sigma \rightarrow \mathcal{P}_f(\mathbb{P}_\Sigma)$:

$$\mathcal{H}(P) := \{ \langle \mathcal{E}_P, \leq_P, \lambda_P, B \rangle \mid B \subseteq \mathcal{B}_P \}.$$

From the definitions it is straightforward to show that $P \sqsubseteq_b Q$ iff Q is isomorphic to some $P' \in \mathcal{H}(P)$. This operator $\mathcal{H}(P)$ can be mirrored on terms; i.e., we can associate inductively to each term s a finite set of terms $\mathbf{H}(s)$ such that $\forall P \in \mathcal{H}(\llbracket s \rrbracket), \exists t \in \mathbf{H}(s) : \llbracket t \rrbracket \cong P$ and $\forall t \in \mathbf{H}(s), \text{CMon}_\square \vdash s \leq t$. We may therefore conclude:

$$\begin{aligned} \llbracket s \rrbracket \sqsubseteq_b \llbracket t \rrbracket &\Rightarrow \exists P \in \mathcal{H}(\llbracket s \rrbracket) : \llbracket t \rrbracket \cong P \\ &\Rightarrow \exists t' \in \mathbf{H}(s) : \llbracket t' \rrbracket \cong \llbracket t \rrbracket \\ &\Rightarrow \exists t' \in \mathbf{H}(s) : \text{BiMon}_\square \vdash t' = t \wedge \text{CMon}_\square \vdash s \leq t' \\ &\Rightarrow \text{CMon}_\square \vdash s \leq t' = t. \end{aligned} \quad \blacktriangleleft$$

We now prove the second half of the theorem:

► **Lemma 38.** *If $\llbracket s \rrbracket \sqsubseteq_o \llbracket t \rrbracket$ then $\text{CMon}_\square \vdash s \leq t$.*

► **Remark 39.** The proof we give below relies on Gischer's completeness theorem [8]. In the Coq proof however, we make no assumptions, and we do not have access to Gischer's result. Therefore we perform a different, more technically involved proof there, with Gischer's theorem as a corollary.

Proof. We will perform this proof by induction on the number of boxes in s . Let $\phi : \llbracket t \rrbracket \rightarrow \llbracket s \rrbracket$ be the box-reflecting homomorphism witnessing $\llbracket s \rrbracket \sqsubseteq_o \llbracket t \rrbracket$.

If s contains no boxes, then by Gischer's completeness theorem we know that

$$\llbracket s \rrbracket \sqsubseteq_o \llbracket t \rrbracket \Rightarrow \text{BiMon}_\square, (\mathbf{B}_1) \vdash s \leq t.$$

Hence, as we have $\text{BiMon}_\square, (\mathbf{B}_1) \subseteq \text{CMon}_\square$, we get $\text{CMon}_\square \vdash s \leq t$.

If on the other hand s has boxes, consider the following set of boxes:

$$\mathcal{B} := \{ B \in \mathcal{B}_{\llbracket s \rrbracket} \mid \forall C \in \mathcal{B}_{\llbracket s \rrbracket}, B \subseteq C \Rightarrow B = C \} (= \max \mathcal{B}_{\llbracket s \rrbracket}).$$

Notice that since ϕ is box-reflecting, we have that $\mathcal{B} = \phi(\max \mathcal{B}_{\llbracket t \rrbracket})$. Furthermore, we have the following property: for any box $B \in \max \mathcal{B}_{\llbracket t \rrbracket}$, the map $\phi|_B$ is a box-reflecting homomorphism from $\llbracket t \rrbracket|_B$ to $\llbracket s \rrbracket|_{\phi(B)}$. We pick new symbols for the elements of \mathcal{B} ; i.e., we find a set Σ' disjoint from Σ and a bijection $\ell(_) : \mathcal{B} \rightarrow \Sigma'$. Using the observation (A.2) we made earlier, we find two maps $s(_), t(_) : \Sigma'$ such that:

$$\forall B \in \mathcal{B}, \quad \text{BiMon}_\square \vdash \pi_B(s) = [s(\ell(B))]. \quad (\text{A.3})$$

$$\forall x \in \Sigma', \quad \mathcal{E}_{\llbracket s(x) \rrbracket} \notin \mathcal{B}_{\llbracket s(x) \rrbracket}. \quad (\text{A.4})$$

$$\forall B \in \max \mathcal{B}_{\llbracket t \rrbracket}, \quad \text{BiMon}_\square \vdash \pi_B(t) = [t(\ell(\phi(B)))]. \quad (\text{A.5})$$

$$\forall x \in \Sigma', \quad \mathcal{E}_{\llbracket t(x) \rrbracket} \notin \mathcal{B}_{\llbracket t(x) \rrbracket}. \quad (\text{A.6})$$

Clearly, for every $x \in \Sigma'$, $s(x)$ has strictly less boxes than s . Our previous observations imply that $\llbracket s(x) \rrbracket \sqsubseteq_o \llbracket t(x) \rrbracket$. Therefore, by induction hypothesis, we get that $\forall x \in \Sigma'$, $\text{CMon}_{\square} \vdash s(x) \leq t(x)$, hence:

$$\forall x \in \Sigma', \text{CMon}_{\square} \vdash [s(x)] \leq [t(x)].$$

Combined with (A.3) and (A.5) this yields: $\forall B \in \max \mathcal{B}_{\llbracket t \rrbracket}$, $\text{CMon}_{\square} \vdash \pi_{\phi(B)}(s) \leq \pi_B(t)$. We define two substitutions $\sigma, \tau : \Sigma \cup \Sigma' \rightarrow \text{SP}_{\Sigma}$ as follows:

$$\sigma(x) := \begin{cases} \pi_B(s) & \text{if } x = \ell(B) \\ x & \text{if } x \in \Sigma \end{cases} \quad \tau(x) := \begin{cases} \pi_B(t) & \text{if } x = \ell(\phi(B)) \\ x & \text{if } x \in \Sigma \end{cases}$$

Finally, we syntactically substitute maximal boxed subterms with letters from Σ' in s, t , yielding terms $s', t' \in \text{SP}_{\Sigma \cup \Sigma'}$ such that $\hat{\sigma}(s') = s$, $\hat{\tau}(t') = t$, and neither s' nor t' has any box. By unfolding the definitions, we can check that since $\llbracket s \rrbracket \sqsubseteq_o \llbracket t \rrbracket$ we have $\llbracket s' \rrbracket \sqsubseteq_o \llbracket t' \rrbracket$. Furthermore, since s' and t' do not contain any box, we may use Gischer's theorem to prove that $\text{CMon}_{\square} \vdash s' \leq t'$. We may now conclude: by applying σ everywhere in the proof of $\text{CMon}_{\square} \vdash s' \leq t'$, we get that $\text{CMon}_{\square} \vdash s = \hat{\sigma}(s') \leq \hat{\sigma}(t')$. Since $\forall a \in \Sigma \cup \Sigma'$ we have $\text{CMon}_{\square} \vdash \sigma(a) \leq \tau(a)$, we get $\text{CMon}_{\square} \vdash \hat{\sigma}(t') \leq \hat{\tau}(t') = t$. \blacktriangleleft

► **Lemma 36.** *Let P, Q be two posets:*

- (i) *if $P \sqsubseteq_b Q$, then P contains \mathbf{P}_1 iff Q contains \mathbf{P}_1 ;*
- (ii) *if $P \sqsubseteq_o Q$, then P contains \mathbf{P}_2 iff Q contains \mathbf{P}_2 ;*
- (iii) *if $P \sqsubseteq_b Q$ and Q contains \mathbf{P}_3 , then P contains \mathbf{P}_3 ;*
- (iv) *if $P \sqsubseteq_b Q$ and Q contains \mathbf{P}_4 , then P contains \mathbf{P}_4 .*

Proof. (i) if $\phi : Q \rightarrow P$ is order reflecting, then by definition we have

$$x \leq_Q y \Leftrightarrow \phi(x) \leq_P \phi(y).$$

The result follows immediately.

(ii) if $\phi : Q \rightarrow P$ is order reflecting, then by definition we have

$$B \in \mathcal{B}_Q \Leftrightarrow \phi(B) \in \mathcal{B}_P$$

$$e \in B \setminus C \Leftrightarrow \phi(e) \in \phi(B) \setminus \phi(C) \quad e \in B \cap C \Leftrightarrow \phi(e) \in \phi(B) \cap \phi(C).$$

The result follows immediately.

(iii) if $\phi : Q \rightarrow P$ is order reflecting and Q contains \mathbf{P}_3 then by definition of the pattern we have $e_1, e_2, e_3 \in \mathcal{E}_Q$ and $B \in \mathcal{B}_Q$ such that

$$e_1 \notin B \quad e_2 \in B \quad e_3 \in B \quad e_1 \leq_Q e_2 \quad e_1 \not\leq_Q e_3.$$

Since ϕ is a poset homomorphism, we know that $\phi(B) \in \mathcal{B}_P$ and $\phi(e_1) \leq_P \phi(e_2)$. By definition of the direct image, we also know that $\phi(e_1) \notin \phi(B)$ and $\phi(e_2), \phi(e_3) \in \phi(B)$. Finally, since ϕ is order reflecting $\phi(e_1) \not\leq_P \phi(e_3)$.

(iv) similar to the proof for iii. \blacktriangleleft

A.3 Proofs related to sets of pomsets

► **Lemma 23.** *There exists a function $T_- : \mathcal{T}_{\Sigma} \rightarrow \mathcal{P}_f(\text{SP}_{\Sigma})$ such that: $\text{SR}_{\square} \vdash e = \sum_{s \in T_e} s$ and $\llbracket e \rrbracket \cong \{\llbracket s \rrbracket \mid s \in T_e\}$.*

Proof. T_e is defined by induction on e :

$$\begin{aligned} T_0 &:= \emptyset & T_1 &:= \{1\} \\ T_a &:= \{a\} & T_{[e]} &:= \{[s] \mid s \in T_e\} \\ T_{e;f} &:= \{s;t \mid \langle s,t \rangle \in T_e \times T_f\} & T_{e\parallel f} &:= \{s\parallel t \mid \langle s,t \rangle \in T_e \times T_f\} \\ T_{e+f} &:= T_e \cup T_f. \end{aligned}$$

Checking the lemma is done by a simple induction. \blacktriangleleft

► **Theorem 24.** *For any pair of terms $e, f \in \mathbf{T}_\Sigma$, the following hold:*

$$\llbracket e \rrbracket \cong \llbracket f \rrbracket \Leftrightarrow \text{SR}_\square \vdash e = f \tag{2.3}$$

$$\llbracket e \rrbracket \downarrow \cong \llbracket f \rrbracket \downarrow \Leftrightarrow \text{CSR}_\square \vdash e = f. \tag{2.4}$$

Proof of (2.3). Soundness is easy to check: by a simple induction on the derivation tree, we can ensure that $\text{SR}_\square \vdash e = f \Rightarrow \llbracket e \rrbracket \cong \llbracket f \rrbracket$.

Using Lemma 23 we may rewrite any term e as a finite union of series-parallel terms. Let $s \in T_e$. By soundness, there is $P \in \llbracket e \rrbracket$ such that $P \cong [s]$. Since $\llbracket e \rrbracket \cong \llbracket f \rrbracket$, there is $Q \in \llbracket f \rrbracket$ such that $P \cong Q$. Since $\text{SR}_\square \vdash f = \sum_{s \in T_f} s$, by soundness there is $t \in T_f$ such that $Q \cong t$. Therefore $[s] \cong [t]$, so by (2.1) we have $\text{BiMon}_\square \vdash s = t$. Since $\text{BiMon}_\square \subseteq \text{SR}_\square$, we also have $\text{SR}_\square \vdash s = t$, and because $t \in T_f$ we get $\text{SR}_\square \vdash s \leq f$. Since this holds for every $s \in T_e$, this means that:

$$\text{SR}_\square \vdash e = \sum_{s \in T_e} s \leq f.$$

By a symmetric argument we obtain $\text{SR}_\square \vdash f \leq e$, allowing us to conclude by antisymmetry that $\text{SR}_\square \vdash e = f$. \blacktriangleleft

Proof of (2.4). Again, soundness is straightforward. For completeness, it is sufficient to show if $\llbracket e \rrbracket \sqsubseteq \llbracket f \rrbracket$ then $\text{CSR}_\square \vdash e \leq f$. Assume $\llbracket e \rrbracket \sqsubseteq \llbracket f \rrbracket$; i.e., every poset in $\llbracket e \rrbracket$ is subsumed by some poset in $\llbracket f \rrbracket$. Since $\text{SR}_\square \subseteq \text{CSR}_\square$, we get by Lemma 23:

$$\text{CSR}_\square \vdash e = \sum_{s \in T_e} s \qquad \text{CSR}_\square \vdash f = \sum_{t \in T_f} t.$$

Let $s \in T_e$, since $\llbracket e \rrbracket \sqsubseteq \llbracket f \rrbracket$, and because of Lemma 23, we know that there is a term $t \in T_f$ such that $[s] \sqsubseteq [t]$. By (2.2), that means $\text{CMon}_\square \vdash s \leq t$. Since $\text{CMon}_\square \subseteq \text{CSR}_\square$, we get $\text{CSR}_\square \vdash s \leq t \leq f$. This means that for every $s \in T_e$, we have $\text{CSR}_\square \vdash s \leq f$, hence that $\text{CSR}_\square \vdash e = \sum_{s \in T_e} s \leq f$. \blacktriangleleft

A.4 Proofs related to Pomset Logic

Proof of (3.2). Let R be a transitive relation, we show by induction on $\Phi \in \mathbf{F}_\Sigma^+$ that $\forall P, Q \in \mathbb{P}_\Sigma$, if $P R Q$ and $Q \models_R \Phi$ then $P \models_R \Phi$.

- If $\Phi = \phi \vee \psi$ or $\Phi = \phi \wedge \psi$, we use the induction hypothesis to show that $Q \models_R \Phi$ implies $P \models_R \Phi$.
- If $\Phi = \perp$, $\Phi = a$, $\Phi = \phi \blacktriangleright \psi$, $\Phi = \phi \star \psi$, $\Phi = [\phi]$, or $\Phi = \langle \phi \rangle$, then the satisfaction relation says $Q \models_R \Phi$ iff $Q R Q'$ and $h(Q')$. Since $P R Q R Q'$, by transitivity of R we get that $P R Q'$ and thus we conclude that $P \models_R \Phi$ without using the induction hypothesis. \blacktriangleleft

Proof of (3.3). Let R be a symmetric and transitive relation, we show by induction on $\Phi \in \mathbf{F}_\Sigma$ that $\forall P, Q \in \mathbb{P}_\Sigma$, if $P R Q$ then $P \models_R \Phi$ iff $Q \models_R \Phi$.

- If $\Phi = \neg\phi$, $\Phi = \phi \vee \psi$, or $\Phi = \phi \wedge \psi$, we use the induction hypothesis to conclude.
- If $\Phi = \perp$, $\Phi = a$, $\Phi = \phi \blacktriangleright \psi$, $\Phi = \phi \star \psi$, $\Phi = [\phi]$, or $\Phi = \langle \phi \rangle$, then the satisfaction relation says $Q \models_R \Phi$ iff $Q \ R \ Q'$ and $h(Q')$. Since $P \ R \ Q$, by symmetry and transitivity of R we get that $P \ R \ Q' \Leftrightarrow Q \ R \ Q'$ and thus we conclude that $P \models_R \Phi \Leftrightarrow Q \models_R \Phi$ without using the induction hypothesis. \blacktriangleleft

Proof of (3.4). This follows by a straightforward induction on formulas. \blacktriangleleft

Proof of (3.5), (3.6), and (3.7). By (3.1), (3.2), and (3.3). \blacktriangleleft

Proof of (3.8). Assume $e \subsetneq f$.

- If $e \models_{\cong}^{\exists} \phi$, then $f \models_{\cong}^{\exists} \phi$:
The first premiss the existence of a poset $P \in \llbracket e \rrbracket \cap \llbracket \phi \rrbracket_{\cong}$. The second allows us to find a poset $Q \in \llbracket f \rrbracket$ such that $P \cong Q$. Since $P \in \llbracket \phi \rrbracket_{\cong}$ and $\llbracket \phi \rrbracket_{\cong}$ is closed under \cong we get $Q \in \llbracket f \rrbracket \cap \llbracket \phi \rrbracket_{\cong}$.
- If $f \models_{\cong}^{\forall} \phi$, then $e \models_{\cong}^{\forall} \phi$:
The premiss tells us that $\llbracket f \rrbracket \subsetneq \llbracket \phi \rrbracket_{\cong}$ and $\llbracket e \rrbracket \subsetneq f$, so by transitivity we get that $\llbracket e \rrbracket \subsetneq \llbracket \phi \rrbracket_{\cong}$; i.e., $e \models_{\cong}^{\forall} \phi$. \blacktriangleleft

Proof of (3.9). Assume $e \sqsubseteq f$.

- If $e \models_{\sqsubseteq}^{\exists} \phi$, then $f \models_{\sqsubseteq}^{\exists} \phi$:
The first premiss the existence of a poset $P \in \llbracket e \rrbracket \cap \llbracket \phi \rrbracket_{\sqsubseteq}$. The second allows us to find a poset $Q \in \llbracket f \rrbracket$ such that $P \sqsubseteq Q$. Since $P \in \llbracket \phi \rrbracket_{\sqsubseteq}$ and $\llbracket \phi \rrbracket_{\sqsubseteq}$ is upwards-closed we get $Q \in \llbracket f \rrbracket \cap \llbracket \phi \rrbracket_{\sqsubseteq}$.
- If $f \models_{\sqsubseteq}^{\forall} \phi$, then $e \models_{\sqsubseteq}^{\forall} \phi$:
The premiss tells us that $\llbracket f \rrbracket \sqsubseteq \llbracket \phi \rrbracket_{\sqsubseteq}$ and $\llbracket e \rrbracket \sqsubseteq f$, so by transitivity we get that $\llbracket e \rrbracket \sqsubseteq \llbracket \phi \rrbracket_{\sqsubseteq}$; i.e., $e \models_{\sqsubseteq}^{\forall} \phi$. \blacktriangleleft

► **Lemma 28.** For any term $s \in \text{SP}_{\Sigma}$ and any poset P , we have:

$$P \models_{\cong} \phi(s) \Leftrightarrow P \cong \llbracket s \rrbracket \quad P \models_{\sqsubseteq} \phi(s) \Leftrightarrow P \sqsupseteq \llbracket s \rrbracket \quad P \models_{\sqsubseteq} \phi(s) \Leftrightarrow P \sqsubseteq \llbracket s \rrbracket.$$

For a term $e \in \text{T}_{\Sigma}$ and a set of posets $X \subseteq \mathbb{P}_{\Sigma}$, we have:

$$X \models_{\cong}^{\forall} \Phi(e) \Leftrightarrow X \subsetneq \llbracket e \rrbracket \quad X \models_{\sqsubseteq}^{\forall} \Phi(e) \Leftrightarrow X \sqsubseteq \llbracket e \rrbracket.$$

Proof. By induction on s :

$$\begin{aligned}
s = 1 : & & P \models_{\cong} \phi(1) = \perp & \Leftrightarrow P \cong \mathfrak{e} = \llbracket 1 \rrbracket. \\
& & P \models_{\sqsubseteq} \phi(1) = \perp & \Leftrightarrow P \cong \mathfrak{e} \Leftrightarrow P \sqsupseteq \mathfrak{e} = \llbracket 1 \rrbracket. \\
& & P \models_{\sqsubseteq} \phi(1) = \perp & \Leftrightarrow P \cong \mathfrak{e} \Leftrightarrow P \sqsubseteq \mathfrak{e} = \llbracket 1 \rrbracket. \\
s = a : & & P \models_{\cong} \phi(a) = a & \Leftrightarrow P \cong \mathfrak{a} = \llbracket a \rrbracket. \\
& & P \models_{\sqsubseteq} \phi(a) = a & \Leftrightarrow P \sqsupseteq \mathfrak{a} = \llbracket a \rrbracket. \\
& & P \models_{\sqsubseteq} \phi(a) = a & \Leftrightarrow P \sqsubseteq \mathfrak{a} = \llbracket a \rrbracket. \\
s = [t] : & & P \models_{\cong} [\phi(t)] & \Leftrightarrow P \cong [Q] \wedge Q \models_{\cong} \phi(t) \\
& & & \Leftrightarrow P \cong [Q] \wedge Q \cong \llbracket t \rrbracket \\
& & & \Leftrightarrow P \cong \llbracket [t] \rrbracket = \llbracket \llbracket t \rrbracket \rrbracket. \\
& & P \models_{\sqsubseteq} [\phi(t)] & \Leftrightarrow P \sqsupseteq [Q] \wedge Q \models_{\sqsubseteq} \phi(t) \\
& & & \Leftrightarrow P \sqsupseteq [Q] \wedge Q \sqsupseteq \llbracket t \rrbracket
\end{aligned}$$

$$\begin{aligned}
& \Leftrightarrow P \sqsupseteq \llbracket [t] \rrbracket = \llbracket [t] \rrbracket . \\
P \models_{\sqsubseteq} [\phi(t)] & \Leftrightarrow P \sqsubseteq [Q] \wedge Q \models_{\sqsubseteq} \phi(t) \\
& \Leftrightarrow P \sqsubseteq [Q] \wedge Q \sqsubseteq \llbracket [t] \rrbracket \\
& \Leftrightarrow P \sqsubseteq \llbracket [t] \rrbracket = \llbracket [t] \rrbracket . \\
s = s_1 ; s_2 : & P \models_{\cong} \phi(s_1) \blacktriangleright \phi(s_2) \Leftrightarrow P \cong P_1 \otimes P_2 \\
& \quad \wedge P_1 \models_{\cong} \phi(s_1) \wedge P_2 \models_{\cong} \phi(s_2) \\
& \Leftrightarrow P \cong P_1 \otimes P_2 \wedge P_1 \cong \llbracket [s_1] \rrbracket \wedge P_2 \cong \llbracket [s_2] \rrbracket \\
& \Leftrightarrow P \cong \llbracket [s_1] \rrbracket \otimes \llbracket [s_2] \rrbracket = \llbracket [s_1 ; s_2] \rrbracket . \\
& P \models_{\sqsupseteq} \phi(s_1) \blacktriangleright \phi(s_2) \Leftrightarrow P \sqsupseteq P_1 \otimes P_2 \\
& \quad \wedge P_1 \models_{\sqsupseteq} \phi(s_1) \wedge P_2 \models_{\sqsupseteq} \phi(s_2) \\
& \Leftrightarrow P \sqsupseteq P_1 \otimes P_2 \wedge P_1 \sqsupseteq \llbracket [s_1] \rrbracket \wedge P_2 \sqsupseteq \llbracket [s_2] \rrbracket \\
& \Leftrightarrow P \sqsupseteq \llbracket [s_1] \rrbracket \otimes \llbracket [s_2] \rrbracket = \llbracket [s_1 ; s_2] \rrbracket . \\
& P \models_{\sqsubseteq} \phi(s_1) \blacktriangleright \phi(s_2) \Leftrightarrow P \sqsubseteq P_1 \otimes P_2 \\
& \quad \wedge P_1 \models_{\sqsubseteq} \phi(s_1) \wedge P_2 \models_{\sqsubseteq} \phi(s_2) \\
& \Leftrightarrow P \sqsubseteq P_1 \otimes P_2 \wedge P_1 \sqsubseteq \llbracket [s_1] \rrbracket \wedge P_2 \sqsubseteq \llbracket [s_2] \rrbracket \\
& \Leftrightarrow P \sqsubseteq \llbracket [s_1] \rrbracket \otimes \llbracket [s_2] \rrbracket = \llbracket [s_1 ; s_2] \rrbracket . \\
s = s_1 \parallel s_2 : & P \models_{\cong} \phi(s_1) \star \phi(s_2) \Leftrightarrow P \cong P_1 \oplus P_2 \\
& \quad \wedge P_1 \models_{\cong} \phi(s_1) \wedge P_2 \models_{\cong} \phi(s_2) \\
& \Leftrightarrow P \cong P_1 \oplus P_2 \wedge P_1 \cong \llbracket [s_1] \rrbracket \wedge P_2 \cong \llbracket [s_2] \rrbracket \\
& \Leftrightarrow P \cong \llbracket [s_1] \rrbracket \oplus \llbracket [s_2] \rrbracket = \llbracket [s_1 \parallel s_2] \rrbracket . \\
& P \models_{\sqsupseteq} \phi(s_1) \star \phi(s_2) \Leftrightarrow P \sqsupseteq P_1 \oplus P_2 \\
& \quad \wedge P_1 \models_{\sqsupseteq} \phi(s_1) \wedge P_2 \models_{\sqsupseteq} \phi(s_2) \\
& \Leftrightarrow P \sqsupseteq P_1 \oplus P_2 \wedge P_1 \sqsupseteq \llbracket [s_1] \rrbracket \wedge P_2 \sqsupseteq \llbracket [s_2] \rrbracket \\
& \Leftrightarrow P \sqsupseteq \llbracket [s_1] \rrbracket \oplus \llbracket [s_2] \rrbracket = \llbracket [s_1 \parallel s_2] \rrbracket . \\
& P \models_{\sqsubseteq} \phi(s_1) \star \phi(s_2) \Leftrightarrow P \sqsubseteq P_1 \oplus P_2 \\
& \quad \wedge P_1 \models_{\sqsubseteq} \phi(s_1) \wedge P_2 \models_{\sqsubseteq} \phi(s_2) \\
& \Leftrightarrow P \sqsubseteq P_1 \oplus P_2 \wedge P_1 \sqsubseteq \llbracket [s_1] \rrbracket \wedge P_2 \sqsubseteq \llbracket [s_2] \rrbracket \\
& \Leftrightarrow P \sqsubseteq \llbracket [s_1] \rrbracket \oplus \llbracket [s_2] \rrbracket = \llbracket [s_1 \parallel s_2] \rrbracket .
\end{aligned}$$

Recall that since $_ \uparrow$ and $_ \downarrow$ are Kuratowski closure operators, they distribute over unions. We may thus obtain:

$$\begin{aligned}
\llbracket \Phi(e) \rrbracket_{\cong} &= \bigcup_{s \in T_e} \llbracket \phi(s) \rrbracket_{\cong} \cong \bigcup_{s \in T_e} \{ \llbracket [s] \rrbracket \} \cong \llbracket [e] \rrbracket . \\
\llbracket \Phi(e) \rrbracket_{\sqsubseteq} &= \bigcup_{s \in T_e} \llbracket \phi(s) \rrbracket_{\sqsubseteq} \cong \bigcup_{s \in T_e} \{ \llbracket [s] \rrbracket \} \downarrow \cong \llbracket [e] \rrbracket \downarrow .
\end{aligned}$$

The statements then follow by (3.5) and (3.6). ◀

Proof of (3.10).

$$\begin{aligned}
e \models_{\cong}^{\exists} \phi(s) & \Leftrightarrow \llbracket [e] \rrbracket \cap \llbracket \phi(s) \rrbracket_{\cong} \neq \emptyset \Leftrightarrow \llbracket [e] \rrbracket \cap \{ \llbracket [s] \rrbracket \} \neq \emptyset \Leftrightarrow \llbracket [s] \rrbracket \in \llbracket [e] \rrbracket . \\
e \models_{\sqsupseteq}^{\exists} \phi(s) & \Leftrightarrow \llbracket [e] \rrbracket \cap \llbracket \phi(s) \rrbracket_{\sqsupseteq} \neq \emptyset \Leftrightarrow \llbracket [e] \rrbracket \cap \{ \llbracket [s] \rrbracket \} \uparrow \neq \emptyset \Leftrightarrow \llbracket [s] \rrbracket \in \llbracket [e] \rrbracket \downarrow .
\end{aligned}$$
◀

► **Proposition 29.** For a pair of series-parallel terms $s, t \in \text{SP}_\Sigma$,

$$\text{BiMon}_\square \vdash s = t \Leftrightarrow \forall \phi \in \mathbf{F}_\Sigma, (\llbracket s \rrbracket \models_\cong \phi \Leftrightarrow \llbracket t \rrbracket \models_\cong \phi) \quad (3.11)$$

$$\text{CMon}_\square \vdash s \leq t \Leftrightarrow \forall \phi \in \mathbf{F}_\Sigma^+, (\llbracket s \rrbracket \models_\sqsupseteq \phi \Rightarrow \llbracket t \rrbracket \models_\sqsupseteq \phi). \quad (3.12)$$

Proof.

$$\text{BiMon}_\square \vdash s = t \Leftrightarrow \llbracket s \rrbracket \cong \llbracket t \rrbracket \quad (2.1)$$

$$\Rightarrow \forall \phi \in \mathbf{F}_\Sigma, \llbracket s \rrbracket \models_\cong \phi \Leftrightarrow \llbracket t \rrbracket \models_\cong \phi. \quad (\text{Lemma 3.3})$$

$$\Rightarrow \llbracket s \rrbracket \models_\cong \phi(t) \quad (\text{Since } \llbracket t \rrbracket \models_\cong \phi(t) \text{ follows from Lemma 28})$$

$$\Leftrightarrow \llbracket s \rrbracket \cong \llbracket t \rrbracket \quad (\text{Lemma 28})$$

$$\Leftrightarrow \text{BiMon}_\square \vdash s = t \quad (2.1)$$

$$\text{CMon}_\square \vdash s \leq t \Leftrightarrow \llbracket s \rrbracket \sqsubseteq \llbracket t \rrbracket \quad (2.2)$$

$$\Rightarrow \forall \phi \in \mathbf{F}_\Sigma^+, \llbracket s \rrbracket \models_\sqsupseteq \phi \Rightarrow \llbracket t \rrbracket \models_\sqsupseteq \phi. \quad (\text{Lemma 3.2})$$

$$\Rightarrow \llbracket t \rrbracket \models_\sqsupseteq \phi(s) \quad (\text{Since } \llbracket s \rrbracket \models_\sqsupseteq \phi(s) \text{ follows from Lemma 28})$$

$$\Leftrightarrow \llbracket t \rrbracket \sqsupseteq \llbracket s \rrbracket \quad (\text{Lemma 28})$$

$$\Leftrightarrow \text{CMon}_\square \vdash s \leq t. \quad (2.2)$$

◀

► **Proposition 30.** Given two terms $e, f \in \mathbf{T}_\Sigma$, the following equivalences hold:

$$\text{SR}_\square \vdash e \leq f \Leftrightarrow (\forall \phi, e \models_{\cong}^{\exists} \phi \Rightarrow f \models_{\cong}^{\exists} \phi) \Leftrightarrow (\forall \phi, f \models_{\cong}^{\forall} \phi \Rightarrow e \models_{\cong}^{\forall} \phi) \quad (3.13)$$

$$\text{SR}_\square \vdash e = f \Leftrightarrow (\forall \phi, e \models_{\cong}^{\exists} \phi \Leftrightarrow f \models_{\cong}^{\exists} \phi) \Leftrightarrow (\forall \phi, e \models_{\cong}^{\forall} \phi \Leftrightarrow f \models_{\cong}^{\forall} \phi) \quad (3.14)$$

$$\text{CSR}_\square \vdash e \leq f \Leftrightarrow (\forall \phi, e \models_{\sqsupseteq}^{\exists} \phi \Rightarrow f \models_{\sqsupseteq}^{\exists} \phi) \Leftrightarrow (\forall \phi, f \models_{\sqsupseteq}^{\forall} \phi \Rightarrow e \models_{\sqsupseteq}^{\forall} \phi) \quad (3.15)$$

$$\text{CSR}_\square \vdash e = f \Leftrightarrow (\forall \phi, e \models_{\sqsupseteq}^{\exists} \phi \Leftrightarrow f \models_{\sqsupseteq}^{\exists} \phi) \Leftrightarrow (\forall \phi, e \models_{\sqsupseteq}^{\forall} \phi \Leftrightarrow f \models_{\sqsupseteq}^{\forall} \phi). \quad (3.16)$$

Proof. (3.13) We prove both directions:

(\Rightarrow) Assume $\text{SR}_\square \vdash e \leq f$. By Corollary 2.3 this means $\llbracket e \rrbracket \sqsubset \llbracket f \rrbracket$. Therefore, we may conclude by (3.8) and (3.8).

(\Leftarrow) We show that each LHS implies $\llbracket e \rrbracket \sqsubset \llbracket f \rrbracket$; i.e., $\text{SR}_\square \vdash e \leq f$:

- Assume $\forall \phi, f \models_{\cong}^{\forall} \phi \Rightarrow e \models_{\cong}^{\forall} \phi$. Then in particular, since $\llbracket f \rrbracket \sqsubset \llbracket f \rrbracket$ by Lemma 28 we have $f \models_{\cong}^{\forall} \Phi(f)$, hence $e \models_{\cong}^{\forall} \Phi(f)$ ergo $\llbracket e \rrbracket \sqsubset \llbracket f \rrbracket$.

- Assume $\forall \phi, e \models_{\cong}^{\exists} \phi \Rightarrow f \models_{\cong}^{\exists} \phi$, and let $P \in \llbracket e \rrbracket$. By Lemmas 23 we know that there is $s \in T_e$ such that $P \cong \llbracket s \rrbracket$, and by Lemma 28 we get $e \models_{\cong}^{\exists} \phi(s)$, hence $f \models_{\cong}^{\exists} \phi(s)$. Hence by (3.10) we get $P \cong \llbracket s \rrbracket \in \llbracket f \rrbracket$.

(3.14) follows from (3.13), and the fact that \leq is antisymmetric. ◀

► **Proposition 31 (Frame properties).** If $P \#_\cong \phi$, and $Q \models_\cong [\phi]$, then it holds that:

(i) $\forall \psi \in \mathbf{F}_\Sigma, P \models_\cong \psi \Leftrightarrow P \oplus Q \models_\cong \psi \star [\phi]$;

(ii) $\forall \psi \in \mathbf{F}_\Sigma, P \models_\cong \psi \Leftrightarrow P \otimes Q \models_\cong \psi \blacktriangleright [\phi]$;

(iii) $\forall \psi \in \mathbf{F}_\Sigma, P \models_\cong \psi \Leftrightarrow Q \otimes P \models_\cong [\phi] \blacktriangleright \psi$.

Proof. Clearly, if $P \models_\cong \psi$, since $Q \models_\cong [\phi]$ we have immediately that $P \oplus Q \models_\cong \psi \star [\phi]$, $P \otimes Q \models_\cong \psi \blacktriangleright [\phi]$, and $P \otimes Q \models_\cong [\phi] \blacktriangleright \psi$.

Now assume $P \oplus Q \models_\cong \psi \star [\phi]$. This means that there exists P', Q' such that $P \oplus Q \cong P' \oplus Q'$, $P' \models_\cong \psi$ and $Q' \models_\cong [\phi]$. The factorization $P \oplus Q \cong P' \oplus Q'$ may be further

decomposed into pomsets P_1, P_2, Q_1 , and Q_2 such that $P \cong P_1 \oplus P_2$, $Q \cong Q_1 \oplus Q_2$, $P_1 \oplus Q_1 \cong P'$, and $P_2 \oplus Q_2 \cong Q'$.

Note that ϕ cannot be satisfied by the empty pomset: otherwise, since the empty pomset is contained in any other pomset, we would have $P \models_{\cong} ([\phi])$. Therefore since both Q' and Q satisfy $[\phi]$, both have to be non-empty boxes. As a result, we get:

- since $Q \cong Q_1 \oplus Q_2$, either Q_1 or Q_2 have to be empty;
- since $Q' \cong P_2 \oplus Q_2$, either P_2 or Q_2 have to be empty.

If Q_2 is empty, we violate again the hypothesis that P is independent from ϕ , since $P \cong P_1 \oplus P_2 \cong P_1 \oplus Q' \ni Q'$, and $Q' \models_{\cong} [\phi]$.

Hence we know that Q_1 and P_2 are empty, meaning that $P \cong P_1 \cong P' \models_{\cong} \psi$.

The same argument works for the sequential product. ◀

► **Note 40.** Counterexamples to Proposition 31:

- For \sqsubseteq , consider the following:

$$P := \mathfrak{a} \qquad Q := [\mathfrak{b} \oplus \mathfrak{c}] \qquad \phi := \langle \mathfrak{c} \rangle \qquad \psi := \mathfrak{a} \star \mathfrak{b}.$$

We may check that in this case we have:

$$P \not\models_{\sqsubseteq} ([\phi]) \qquad Q \models_{\sqsubseteq} [\phi] \qquad P \oplus Q \models_{\sqsubseteq} \psi \star [\phi] \qquad P \not\models_{\sqsubseteq} \psi.$$

This happens because in order to satisfy $\psi \star [\phi]$ we may rearrange the pomsets using the ordering. More precisely we use the fact that:

$$P \oplus Q = \mathfrak{a} \oplus [\mathfrak{b} \oplus \mathfrak{c}] \sqsubseteq (\mathfrak{a} \oplus \mathfrak{b}) \oplus \mathfrak{c}.$$

- For \sqsupseteq , we have a similar example. The difference is that instead of Q overspiling into P , we have the converse.

$$P := \mathfrak{a} \oplus \mathfrak{b} \qquad Q := \mathfrak{c} \qquad \phi := \langle \mathfrak{c} \rangle \qquad \psi := \mathfrak{a}.$$

Since P does not contain any c , but has more than one event, $P \not\models_{\sqsubseteq} ([\phi]) \vee \psi$. On the other hand $Q \models_{\sqsubseteq} [\phi]$ because $Q \sqsupseteq [Q]$ and $Q \ni Q \models_{\sqsupseteq} \mathfrak{c}$. Finally, $P \oplus Q = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{c} \sqsupseteq \mathfrak{a} \oplus [\mathfrak{b} \oplus \mathfrak{c}]$, and we have both $\mathfrak{a} \models_{\sqsubseteq} \psi$ and $[\mathfrak{b} \oplus \mathfrak{c}] \models_{\sqsubseteq} [\phi]$.