Proof-theoretic Semantics in Sheaves
(Extended Abstract)

David Pym — University College London
Eike Ritter — University of Birmingham
Edmund Robinson — Queen Mary University of London

In proof-theoretic semantics [6], model-theoretic validity is replaced by proof-theoretic validity. Validity of formulae is defined inductively from a base giving the validity of atoms using inductive clauses derived from proof-theoretic rules. A key aim is to show completeness of the proof rules without any requirement for formal models. Establishing this for propositional intuitionistic logic (IPL) raises some technical and conceptual issues [2, 3, 5].

We relate the (complete) base-extension semantics of [5] to categorical proof theory and sheaf-theoretic semantics (e.g., [1]). For the latter, propositions are interpreted as functors from a category of bases to the lattice $\{\{\top\}, \emptyset\}$. This set of functors forms the truth values of a topos of functors from bases to **Set**. There are two critical aspects: the stability of interpretation under extension of bases lands us in the world of Kripke models, and the non-standard interpretation of disjunction is revealed to come from a Grothendieck topology.

Base-extension Semantics in Presheaves. Sandqvist [5] gives a base-extension proof-theoretic semantics for IPL for which natural deduction is sound and complete. A base $\mathcal{B}$ is a set of atomic rules (for $\vdash_{\mathcal{B}}$) as in Definition 1, which also defines the application of base rules, and satisfaction in a base ($\Vdash_{\mathcal{B}}$). Roman p, P, etc. denote atoms and sets of atoms; Greek ϕ, Γ, etc. denote formulae and sets of formulae.

Definition 1 (Sandqvist's Semantics) *Base rules $\mathcal{R}$, application of base rules, and satisfaction of formulae in a (possibly finite) countable base $\mathcal{B}$ of rules $\mathcal{R}$ are defined as follows:*

$$\frac{\begin{matrix}[P_1] & & [P_n] \\ q_1 & \dots & q_n\end{matrix}}{r}\ \mathcal{R}$$

(Ref) $P, p \vdash_{\mathcal{B}} p$

(App$_{\mathcal{R}}$) if $((P_1 \Rightarrow q_1), \dots, (P_n \Rightarrow q_n)) \Rightarrow r)$ and, for all $i \in [1, n]$, $P, P_i \vdash_{\mathcal{B}} q_i$, then $P \vdash_{\mathcal{B}} r$

(At) for atomic p, $\Vdash_{\mathcal{B}} p$ iff $\vdash_{\mathcal{B}} p$

($\supset$) $\Vdash_{\mathcal{B}} \phi \supset \psi$ iff $\phi \Vdash_{\mathcal{B}} \psi$

($\wedge$) $\Vdash_{\mathcal{B}} \phi \wedge \psi$ iff $\Vdash_{\mathcal{B}} \phi$ and $\Vdash_{\mathcal{B}} \psi$

($\vee$) $\Vdash_{\mathcal{B}} \phi \vee \psi$ iff, for every atomic p and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi \Vdash_{\mathcal{C}} p$ and $\psi \Vdash_{\mathcal{C}} p$, then $\Vdash_{\mathcal{C}} p$

($\bot$) $\Vdash_{\mathcal{B}} \bot$ iff, for all atomic p, $\Vdash_{\mathcal{B}} p$

(Inf) for $\Theta \neq \emptyset$, $\Theta \Vdash_{\mathcal{B}} \phi$ iff, for every $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \theta$ for every $\theta \in \Theta$, then $\Vdash_{\mathcal{C}} \phi$

There is a substitution (cut) operation on bases that maps derivations $P \vdash_{\mathcal{B}} p$ and $p, Q \vdash_{\mathcal{B}} q$ to a derivation $P, Q \vdash_{\mathcal{B}} q$.

Key to understanding our categorical formulation is the Yoneda lemma (see [1]): let $\mathcal{C}$ be a locally small category, let **Set** be the category of sets, and $F \in [\mathcal{C}^{op}, \mathbf{Set}]$ (the category of presheaves over $\mathcal{C}$); then, for each object C of $\mathcal{C}$, with $h^C = \hom(-, C)$, the natural transformations $\mathrm{Nat}(h^C, F) \equiv \hom(\hom(-, C), F) \cong F(C)$.

We give a category-theoretic formulation of proof-theoretic validity using presheaves (i.e., functors $F \in [\mathcal{W}^{op}, \mathbf{Set}]$), where $\mathcal{W}$ has objects pairs $(\mathcal{B}, P)$ and morphisms are given by coinclusions of the base and derivations in the larger base. Composition is given by substitution.

Define a functor $[\![\phi]\!] \colon \mathcal{W}^{op} \to \mathbf{Set}$ by induction over the structure of ϕ as follows: the base case $[\![p]\!](\mathcal{B}, P)$ is the set of derivations $P \vdash_{\mathcal{B}} p$. $[\![p]\!]$ applied to morphisms is given by substitution. The definition is extended to the connectives homomorphically. A key step is the use of the Yoneda lemma to define the (hom-set) interpretation of $\supset$, which is used to define the interpretation of Sandqvist's (elimination-style) semantics for $\vee$ (see also below). Thus we establish the formal functoriality and naturality of Sandqvist's semantics.

Theorem 2 (Soundness & Completeness) *Define (cf. [5])* $\Gamma \Vdash \phi$ *as: for all* $\mathcal{B}$*, if* $\Vdash_{\mathcal{B}} \psi$ *for all* $\psi \in \Gamma$*, then* $\Vdash_{\mathcal{B}} \phi$*. Then* $\Gamma \vdash \phi$ *(in natural deduction for IPL, cf. [5]) iff* $\Gamma \Vdash \phi$*.*

The proof of soundness uses the existence of a natural transformation corresponding to $\Vdash$: $\Gamma \Vdash \phi$ iff there exists a natural transformation from $[\![\Gamma]\!]$ to $[\![\phi]\!]$. The proof of completeness uses a special base, as in [5], which is extended via $[\![-]\!]$ to the full consequence relation.

Sheaves and Disjunction. Standard Kripke semantics interprets both conjunction and disjunction pointwise (i.e., on each base, in proof-theoretic semantics [3]), while it relies on the extension ordering for implication (cf. the discussion of Goldfarb's semantics in [3]). This is a result of the requirement that the set of bases validating any proposition should be closed under extension: propositions do not become untrue if we are given additional atomic information. But there is an issue over the interpretation of disjunction. A standard constructive view is that the proof of a disjunction should resolve to a proof of one of the disjuncts. This is not obviously stable under extension of information and obtaining a pointwise disjunction reflecting this viewpoint is the hardest part of the proof of completeness of standard Kripke models for IPL. We show that Sandqvist's approach avoids this difficulty by using a Grothendieck topology.

In this section, we ignore differences between derivations, and interpret propositions as truth values in the topos $\mathcal{S} = [\mathcal{W}^{op}, \mathbf{Set}]$. These can be identified with subfunctors of the constant singleton functor $\{\top\}$ (cf. [1]). Atomic propositions are interpreted in $\mathcal{S}$ via $[\![p]\!](\mathcal{B}, P) = \{\top \mid P \vdash_{\mathcal{B}} p\} = \{\top \mid P \Vdash_{\mathcal{B}} p\}$. Sandqvist's satisfaction conditions for conjunction and implication correspond to the internal interpretation of the logic in the topos $\mathcal{S}$, but his conditions for disjunction and false do not.

For each atomic proposition p, we form an internal operator on truth values: $j_p(\omega) = (\omega \supset [\![p]\!]) \supset [\![p]\!]$. The set of atomic propositions internalizes as the constant functor: $At(\mathcal{B}) = \{p \mid p \text{ is atomic}\}$. Consider the function on truth values that is the internal interpretation of $j(\omega) = \forall p \in At.\ j_p(\omega) = \forall p \in At.\ (\omega \supset [\![p]\!]) \supset [\![p]\!]$. This is a Lawvere-Tierney topology — that is, the internalization of a Grothendieck topology — and each $[\![p]\!]$ is j-closed.

Sandqvist's satisfaction conditions correspond exactly to the standard interpretation of connectives in the topos of sheaves for this topology.

Proposition 3 *For any proposition* ϕ*, and any world* $W = (\mathcal{B}, P)$*,* $P \Vdash_{\mathcal{B}} \phi$ *iff* $[\![\phi]\!](\mathcal{B}, P) = \{\top\}$*, where* $[\![\phi]\!]$ *is the standard interpretation of* ϕ *in* $Sh_j(\mathcal{S})$*.*

This follows from the closure of sheaves under conjunction and implication, intuitionistic equivalence of $((\phi \vee \psi) \supset p) \supset p$ and $((\phi \supset p) \wedge (\psi \supset p)) \supset p$, and expansion of definitions.

This sheaf model can be seen as a continuation semantics in which a complete proof-search [4] is the proof of an atomic proposition. Using a topology for this results in a disjunction being valid iff a point is covered by refinements on each of which one of the disjuncts holds — cf. Beth's semantics (see, e.g., [1]).

References

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