Stochastic Differential Equations

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UCL

Outline

- Summary RG I

- Theory of Stochastic Differential Equations

  - Linear Stochastic Differential Equations
  - Reducible Stochastic Differential Equations
  - Comments on the types of solutions
    - Weak vs Strong
    - Stratonovich SDEs

- Modelling with Stochastic Differential Equations

  (Yuan Shen)

Recap 0

- Continuous time continuous state Markov processes:

$$dx = \alpha(t, x)\, dt$$

$$dX_t = \alpha(t, X_t)\, dt$$

$$dX_t = \alpha(t, X_t)\, dt + \beta(t, X_t)\, dW_t$$

- Random differential equations:
  - Random coefficients (or random initial values)
  - Continuous and differentiable sample paths
  - Solved sample path by sample path (ODE)

- Stochastic differential equations:
  - Random coefficients
  - Continuous, but non-differentiable sample paths *(irregular stochastic processes)*
  - Differentials to be interpreted as stochastic integrals!

Recap I

- A Markov process is a diffusion process if the following limits exist:

$$\lim_{t \downarrow s} \frac{\int p(s,x;t,y)\, dy}{t-s} = 0,$$

$$\lim_{t \downarrow s} \frac{E\{y-x\}}{t-s} = \alpha(s,x),$$ drift

$$\lim_{t \downarrow s} \frac{E\{(y-x)^2\}}{t-s} = \beta^2(s,x).$$ diffusion

- Standard Wiener process:

$$W_0 = 0 \quad \text{w.p. } 1,$$

$$E\{W_t\} = 0,$$

$$W_t - W_s \sim \mathcal{N}(0, t-s).$$

  - Independent Gaussian increments
  - Almost surely continuous (in time) sample paths
  - Almost surely non differentiable

Recap II

- Ito formula (stochastic chain rule):

$$Y_t = U(t, X_t) \qquad dX_t = e\, dt + f\, dW_t$$

$E\{dW_t^2\} \sim \mathcal{O}(dt)$

$$dY_t = \left\{ \frac{\partial U}{\partial t} + e \frac{\partial U}{\partial x} + \frac{1}{2} f_t^2 \frac{\partial^2 U}{\partial x^2} \right\} dt + f \frac{\partial U}{\partial x}\, dW_t.$$

To be interpreted as…

- Ito integral:

$$Y_t - Y_{t_0} = \int_{t_0}^{t} \left\{ \frac{\partial U}{\partial t} + e \frac{\partial U}{\partial x} + \frac{1}{2} f_t^2 \frac{\partial^2 U}{\partial x^2} \right\} dt + \int_{t_0}^{t} f \frac{\partial U}{\partial x}\, dW_t \ \text{ w.p. 1.}$$

  - Martingale property
  - Zero-mean random variable
  - Equality in mean square sense

Linear stochastic differential equations

$$dX_t = \alpha(t, X_t)\, dt + \beta(t, X_t)\, dW_t$$

$$\alpha(t, X_t) = a_1(t)X_t + a_2(t) \qquad \beta(t, X_t) = b_1(t)X_t + b_2(t)$$

- The linear SDE is autonomous if all coefficients are constants.
- The linear SDE is homogeneous if $a_2(t) = 0, \quad b_2(t) = 0.$
- The SDE is linear in the narrow sense (additive noise) if $b_1(t) = 0.$
- The noise is multiplicative if $b_2(t) = 0.$

$$\int_0^t \beta X_t\, dW_t \qquad \int_0^t \beta\, dW_t$$

General solution to a linear SDE in the narrow sense

$$dX_t = \{a_1(t)X_t + a_2(t)\}\, dt + b_2(t)\, dW_t$$

- Fundamental (or homogeneous) solution:

$$d(\ln X_t) = a_1(t)\, dt \quad \Rightarrow \quad \Phi_{t,t_0} = e^{\int_{t_0}^{t} a_1(s)\, ds}$$

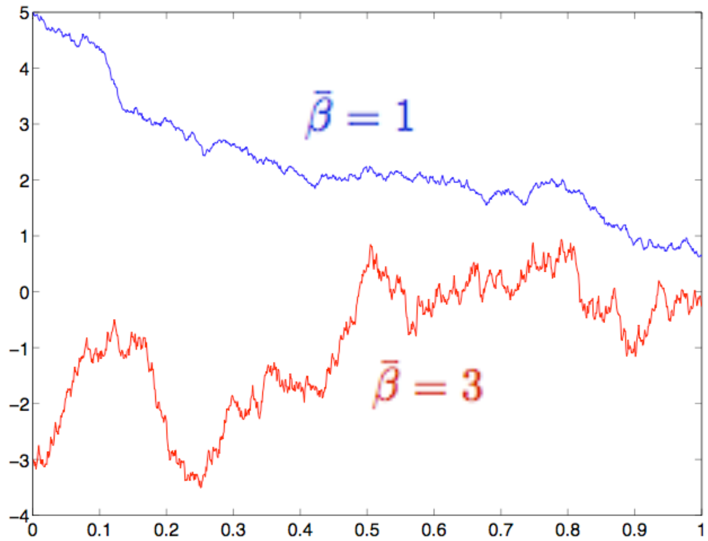
- Applying the Ito formula leads to

$$\left.\begin{aligned} U(t,x) &= \Phi_{t,t_0}^{-1} x \\ Y_t &= U(t, X_t) \end{aligned}\right\} \Rightarrow dY_t = a_2(t)\Phi_{t,t_0}^{-1}\, dt + b_2(t)\Phi_{t,t_0}^{-1}\, dW_t$$

- The integral form is given by

$$X_t = \Phi_{t,t_0}\left\{X_{t_0} + \int_{t_0}^{t} a_2(s)\Phi_{s,t_0}^{-1}\, ds + \int_{t_0}^{t} b_2(s)\Phi_{s,t_0}^{-1}\, dW_s\right\}$$

Example: Langevin equation



$$dX_t = -\bar{\alpha} X_t \, dt + \bar{\beta} \, dW_t$$

- Autonomous linear SDE in the narrow sense:

$$\begin{cases} a_1(t) = -\bar{\alpha}, \\ a_2(t) = 0, \\ b_1(t) = 0, \\ b_2(t) = \bar{\beta}. \end{cases}$$

- Solution:

$$X_t = e^{-\bar{\alpha}(t-t_0)} X_{t_0} + e^{-\bar{\alpha}(t-t_0)} \int_{t_0}^{t} \bar{\beta} e^{\bar{\alpha}(s-t_0)} \, dW_s$$

- Homogeneous Gaussian process

- Means: $E\{X_t\} = e^{-\bar{\alpha} t} E\{X_0\}$

- Variances:

$$E\{X_t^2\} = e^{-2\bar{\alpha} t} E\{X_0^2\} + \tfrac{\bar{\beta}^2}{2\bar{\alpha}} (1 - e^{-2\bar{\alpha} t})$$

$$\Rightarrow \quad V\{X_t\} = e^{-2\bar{\alpha} t} V\{X_0\} + \tfrac{\bar{\beta}^2}{2\bar{\alpha}} (1 - e^{-2\bar{\alpha} t})$$

General solution to a linear SDE

$$dX_t = \{a_1(t)X_t + a_2(t)\}\, dt + \{b_1(t)X_t + b_2(t)\}\, dW_t$$

- Fundamental solution:

$$d(\ln X_t) = \left\{a_1(t) - \tfrac{1}{2}b_1^2(t)\right\}\, dt + b_1(t)\, dW_t$$

$$\Rightarrow \quad \Phi_{t,t_0} = e^{\int_{t_0}^t (a_1(s) - \frac{1}{2}b_1^2(s))\, ds + \int_{t_0}^t b_2(s)\, dW_t}$$

- Apply Ito formula to compute:

$$d(\Phi_{t,t_0}^{-1}) \qquad dY_t \quad \text{with} \quad \begin{cases} Y_t = U(t, X_t, \Phi_{t,t_0}^{-1}) \\ U(t, x_1, x_2) = x_1 x_2 \end{cases}$$

- Integral form of the solution:

$$X_t = \Phi_{t,t_0}\left\{X_{t_0} + \int_{t_0}^t (a_2(s) + b_1(s)b_2(s))\Phi_{s,t_0}^{-1}\, ds + \int_{t_0}^t b_2(s)\Phi_{s,t_0}^{-1}\, dW_s\right\}$$

Ordinary differential equations for the first two moments

$$dX_t = \{a_1(t)X_t + a_2(t)\}\, dt + \{b_1(t)X_t + b_2(t)\}\, dW_t$$

- ODE of the means:

$$\frac{dm}{dt} = a_1 m + a_2$$

- ODE of the second order moment:

$$\frac{dP}{dt} = (2a_1 + b_1^2)P + 2(a_2 + b_1 b_2)m + b_2^2$$

Reducible Stochastic Differential Equations

- Idea:

$$dY_t = \alpha(t, Y_t)\ dt + \beta(t, Y_t)\ dW_t$$

$$\Downarrow U(t, Y_t) = X_t\ ?$$

$$dX_t = (a_1(t)X_t + a_2(t))\ dt + (b_1(t)X_t + b_2(t))\ dW_t$$

- Conditions:

$$\begin{cases} a_1 U + a_2 = \frac{\partial U}{\partial t} + \alpha \frac{\partial U}{\partial y} + \frac{1}{2}\beta^2 \frac{\partial^2 U}{\partial y^2} \\ b_1 U + b_2 = \beta \frac{\partial U}{\partial y} \end{cases}$$

- *Special cases: see Kloeden & Platen (1999), p.115-116.*

Types of solutions

- Under some regularity conditions on α and β, the solution to the SDE

$$dX_t = \alpha(t, X_t)\, dt + \beta(t, X_t)\, dW_t$$

  is a diffusion process.
- A solution is a strong solution if it is valid for each given Wiener process (and initial value), that is it is sample pathwise unique.

- A diffusion process with its transition density satisfying the Fokker-Planck equation is a solution of a SDE.
- A solution is a weak solution if it is valid for given coefficients, but unspecified Wiener process, that is its probability law is unique.

Comments on Stratonovich SDEs

$$f(t,\omega)$$

$$\downarrow$$

$$f^{(n)}(t,\omega) = f(\tau_j^{(n)},\omega)$$

$$\tau_j^{(n)} = (1-\lambda)t_j^{(n)} + \lambda t_{j+1}^{(n)}$$

- Choice of the definition of a stochastic integral
  - Leads to distinct solutions for same coefficients
  - Solutions (may) behave differently

- Possibility to switch from one interpretation to the other:
  - Ito SDE determines appropriate coefficients for the Fokker-Planck equations
  - Stratonovich obeys rules of classical calculus

Next reading groups…

- Who?
- When?
- Where?

- How?

References

- P. Kloeden and E. Platen: *Numerical Solutions to Stochastic Differential Equations*. Springer-Verlag, 1999 (3rd edition).

- B. Øksendael: *Stochastic Differential Equations*. Springer-Verlag, 2002 (6th edition).

- Lawrence E. Evans. An Introduction to Stochastic Differential Equations. Lecture notes (Department of Mathematics, UC Berkeley), available from http://math.berkeley.edu/ evans/.