An introduction to Lévy processes

with financial modelling in mind

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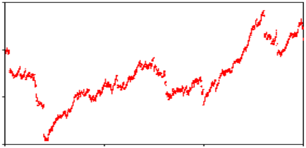

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More recently

- Financial modelling: non-Normal returns, volatility smiles
- Financial modelling: stochastic volatility, leverage effects
- Population and phylogenetic models: trees encoded in Lévy processes

The structure of Lévy processes

Definition (Lévy process)

A Lévy process is a continuous-time stochastic process $(X_t, t \geq 0)$ with

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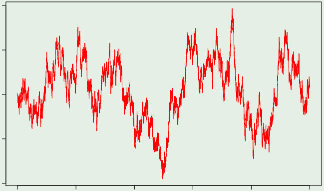
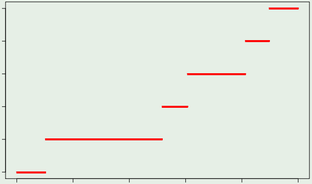
This definition makes sense essentially for processes with values in a locally compact topological group. We will focus on the real-valued case, extensions to k-dimensional Euclidean space are straightforward.

Examples

1. Brownian motion $B_t \sim \mathrm{Normal}(\mu t, \sigma^2 t)$.
2. Poisson process $N_t \sim \mathrm{Poisson}(\lambda t)$ for some intensity $\lambda \in (0, \infty)$.





Their moment generating functions exists, for all $\gamma \in \mathbb{R}$:

$$\mathbb{E}(e^{\gamma B_t}) = \int_{-\infty}^{\infty} e^{\gamma x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} dx = \exp\left\{t\left(\mu\gamma + \frac{1}{2}\sigma^2\gamma^2\right)\right\}$$

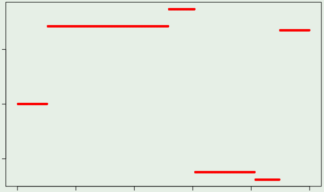
$$\mathbb{E}(e^{\gamma N_t}) = \sum_{n=0}^{\infty} e^{\gamma n} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \exp\left\{t\lambda\left(e^{\gamma} - 1\right)\right\}.$$

Example

3. Compound Poisson process

$$C_t = \sum_{k=1}^{N_t} A_k,$$

- where A_k, $k \geq 1$, independent identically distributed
- and $(N_t, t \geq 1)$ is an independent Poisson process.



Its moment generating function exists if and only if $\mathbb{E}(e^{\gamma A_1})$ exists and then

$$\begin{aligned}\mathbb{E}(e^{\gamma C_t}) &= \sum_{n=0}^{\infty} \mathbb{E}\left(\exp\left\{\gamma \sum_{k=1}^{n} A_k\right\}\right) \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \left(\mathbb{E}\left(e^{\gamma A_1}\right)\right)^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \exp\left\{t\lambda\left(\mathbb{E}\left(e^{\gamma A_1}\right) - 1\right)\right\}.\end{aligned}$$

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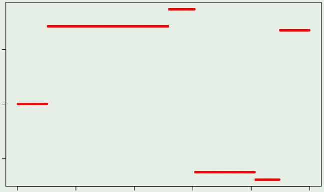

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Definition (Infinite divisibility)

A random variable Y (or its distribution) is called *infinitely divisible* if

$$Y \sim Y_1^{(n)} + \ldots + Y_n^{(n)}$$

for all $n \geq 2$ and independent and identically distributed $Y_1^{(n)}, \ldots, Y_n^{(n)}$.

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Proposition

X_t is infinitely divisible for a Lévy process $(X_t, t \geq 0)$.

Proof.

We recall the stationarity and independence of increments and take $Y_j^{(n)} = X_{jt/n} - X_{(j-1)t/n}$. □

Theorem (Lévy-Khintchine)

A random variable Y is infinitely divisible if and only if

$$\mathbb{E}(e^{i\xi Y}) = \exp\left\{ i\mu\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \left(e^{i\xi x} - 1 - i\xi x 1_{\{|x|\le 1\}} \right) \nu(dx) \right\}$$

for some $\mu \in \mathbb{R}$, $\sigma^2 \ge 0$, and ν on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty$.

Furthermore, if $\mathbb{E}(e^{\gamma Y}) < \infty$, then

$$\mathbb{E}(e^{\gamma Y}) = \exp\left\{ \mu\gamma + \frac{1}{2}\sigma^2\gamma^2 + \int_{\mathbb{R}} \left(e^{\gamma x} - 1 - \gamma x 1_{\{|x|\le 1\}} \right) \nu(dx) \right\}.$$

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In fact, $Y \sim B_1 + C_1 + M_1$, where $\lambda = \nu([-1,1]^c)$, $A_k \sim \lambda^{-1}\nu|_{[-1,1]^c}$ and

$$M_1 = L^2\text{-}\lim_{\varepsilon \downarrow 0} \left(C_1^{(\varepsilon)} - \int_{\mathbb{R}} x 1_{\{\varepsilon < |x| \le 1\}} \nu(dx) \right).$$

Construction of Lévy processes

Every Lévy process is infinitely divisible. For every infinitely divisible distribution there is a Lévy process, the limit as $\varepsilon \downarrow 0$ of

$$X_t^{(\varepsilon)} = \mu t + \sigma W_t + C_t + M_t^{(\varepsilon)}, \qquad M_t^{(\varepsilon)} = C_t^{(\varepsilon)} - t \int_{\mathbb{R}} x 1_{\{\varepsilon < |x| \le 1\}} \nu(dx),$$

where

- $(W_t, t \ge 0)$ is standard Brownian motion with $W_t \sim \mathrm{Normal}(0, t)$,
- $(C_t, t \ge 0)$ is an independent compound Poisson process with $\lambda = \nu((-1,1)^c)$, $A_k \sim \lambda^{-1} \nu|_{[-1,1]^c}$,
- $(C_t^{(\varepsilon)}, t \ge 0)$ is an independent compound Poisson process with $\lambda_\varepsilon = \nu(\varepsilon < |x| \le 1)$, $A_k^{(\varepsilon)} \sim \lambda_\varepsilon^{-1} \nu|_{\{\varepsilon < |x| \le 1\}}$.

Note that $C_t^{(\varepsilon)}$ has jumps of sizes $A_k^{(\varepsilon)} \in \{\varepsilon < |x| \le 1\}$.

For a sequence $a_0 = 1$, $a_n \downarrow 0$, and $I_n = (a_n, a_{n-1}]$, $I_{-n} = [-a_{n-1}, -a_n)$,

$$X_t = \mu t + \sigma W_t + \sum_{n \in \mathbb{Z}} \left(C_t^{[n]} - \mu_n t \right), \qquad \mu_0 = 0, \mu_n = \int_{I_n} x \nu(dx).$$

Convergence of this series means, we can approximate X_t by only considering $I_{-n_0}, \ldots, I_{n_0}$ for sufficiently high n_0.

Is compensation of small jumps necessary? In general, yes.

- However, if $\int_{\mathbb{R}} (|x| \wedge 1) \nu(dx) < \infty$, then we do not need compensation

$$\mathbb{E}(e^{i\xi X_1}) = \exp\left\{ i\mu\,\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \left(e^{i\xi x} - 1 - i\xi x 1_{\{|x| \le 1\}} \right) \nu(dx) \right\};$$

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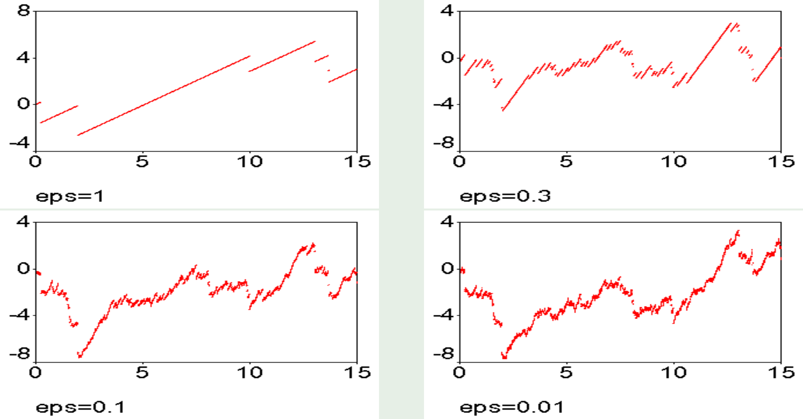
Example (Compensation)

For $\mu'' = \sigma^2 = 0$, $\nu(dx) = |x|^{-5/2} 1_{\{-3 \leq x < 0\}} dx$, the compensating drifts $\int_\varepsilon^3 x\nu(dx)$ are 0.845, 2.496, 5.170 and 18.845 for $\varepsilon = 1$, $\varepsilon = 0.3$, $\varepsilon = 0.1$ and $\varepsilon = 0.01$. In the simulation, the slope increases (to infinity, as $\varepsilon \downarrow 0$):



Method (Simulation by time discretisation)

Fix a lag $\delta > 0$. *Denote* $F_\delta^{-1}(u) = \inf\{x \in \mathbb{R} : \mathbb{P}(X_\delta \le x) > u\}$. *Simulate*

$$X_t^{(1,\delta)} = S_{[t/\delta]}, \qquad \textit{where } S_n = \sum_{k=1}^{n} Y_k \textit{ and } Y_k = F_\delta^{-1}(U_k).$$

Method (Simulation by throwing away small jumps)

Let $\sigma^2 = 0$. *Fix a jump size threshold* $\varepsilon > 0$ *so that* $\lambda_\varepsilon = \nu(|x| > \varepsilon) > 0$. *Denote* $H_\varepsilon^{-1}(u) = \inf\{x \in \mathbb{R} : \lambda_\varepsilon^{-1}\nu((-\infty, x] \cap [-\varepsilon, \varepsilon]^c) > u\}$. *Simulate*

$$N_t = \#\{n \ge 1 : T_n \le t\}, \qquad \textit{where } T_n = \sum_{k=1}^{n} Z_k \textit{ and } Z_k = -\lambda_\varepsilon^{-1} \ln(U_{2k-1}),$$

$$X_t^{(2,\varepsilon)} = S_{N_t} - b_\varepsilon t, \qquad \textit{where } S_n = \sum_{k=1}^{n} A_k \textit{ and } A_k = H_\varepsilon^{-1}(U_{2k}),$$

with $b_\varepsilon = \mu - \int_{\{x \in \mathbb{R} : \varepsilon < |x| \le 1\}} x\nu(dx)$.

Theorem (Asmussen and Rosinski)

Let $(X_t, t \geq 0)$ be a Lévy process with characteristics $(a, 0, \nu)$. Denote $v(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} x^2 \nu(dx)$. If $v(\varepsilon)/\varepsilon^2 \to \infty$ as $\varepsilon \downarrow 0$, then

$$\frac{X_t - X_t^{(2,\varepsilon)}}{\sqrt{v(\varepsilon)}} \to W_t \qquad \text{in distribution as } \varepsilon \downarrow 0$$

for an independent Brownian motion $(W_t)_{t \geq 0}$

If $v(\varepsilon)/\varepsilon^2 \to \infty$, it is well-justified to adjust Method 2 to set

$$X_t^{(2+,\varepsilon)} = X_t^{(2,\varepsilon)} + \sqrt{v(\varepsilon)} W_t$$

for an independent Brownian motion. We thereby approximate the small jumps (that we throw away) by an independent Brownian motion.

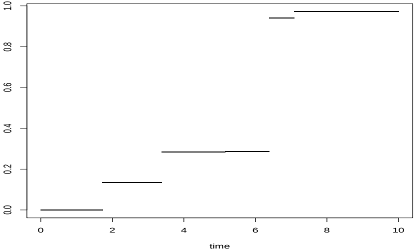
Parametric families

We can construct distributions that are infinitely divisible and consider associated Lévy processes.

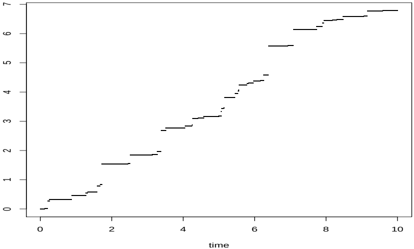
1. Brownian motion $B_t \sim \mathrm{Normal}(\mu t, \sigma^2 t)$, two parameters $\mu \in \mathbb{R}$, $\sigma^2 \geq 0$.
2. Gamma process $G_t \sim \mathrm{Gamma}(\alpha t, \beta)$, two parameters $\alpha > 0$, $\beta > 0$.
3. Generalised Variance Gamma process $V_t = G_t - H_t$ for $G_t \sim \mathrm{Gamma}(\alpha_+ t, \beta_+)$ and $H_t \sim \mathrm{Gamma}(\alpha_- t, \beta_-)$ independent, four parameters $\alpha_\pm > 0$, $\beta_\pm > 0$.
4. Normal Inverse Gaussian $Z_t = B^{(2)}_{I_t}$, $I_t = \inf\{s \geq 0 : B^{(1)}_s > t\}$ for two independent Brownian motions $B^{(i)}_s \sim \mathrm{Normal}(\mu_i s, \sigma_i^2 s)$, three parameters $\mu_1 \geq 0$, $\mu_2 \in \mathbb{R}$, $\sigma_2^2 \geq 0$, w.l.o.g. $\sigma_1^2 = 1$.

Advantages: explicit densities, useful for parameter estimation.
Disadvantages: not much modelling freedom, may lead to poor fit

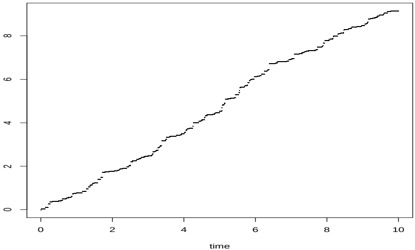
Gamma processes for parameters $\alpha = \beta \in \{0.1, 1, 10, 100\}$.



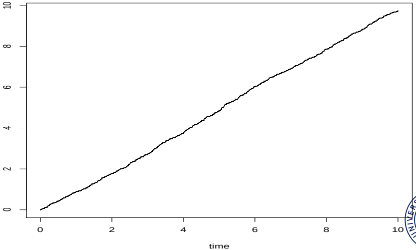
Gamma process with shape parameter 0.1 and scale parameter 0.1



Gamma process with shape parameter 1 and scale parameter 1

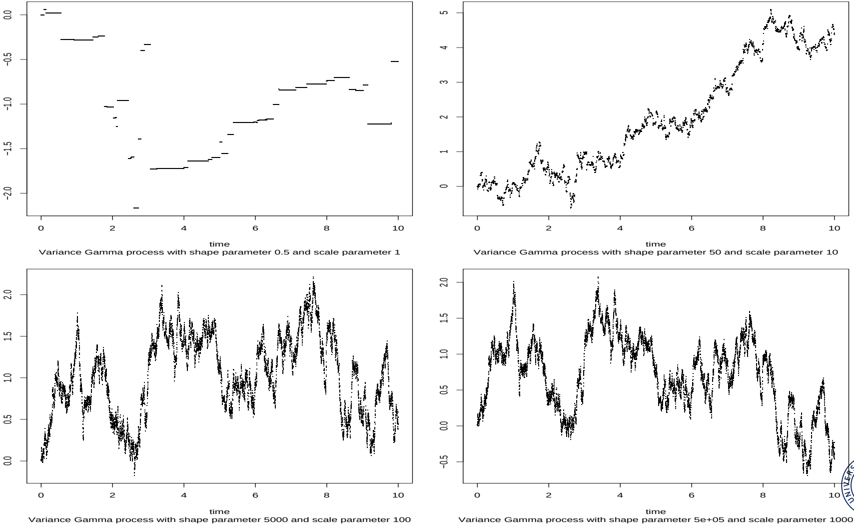


Gamma process with shape parameter 10 and scale parameter 10



Gamma process with shape parameter 100 and scale parameter 100

Variance gamma processes for parameters $\sqrt{2\alpha} = \beta \in \{1, 10, 100, 1000\}$.



More flexibly, we can specify characteristics (μ, σ^2, ν).

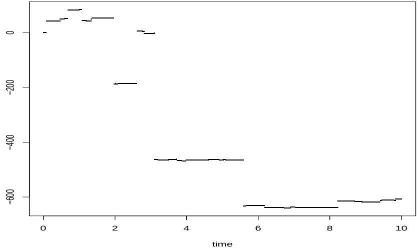
1. Stable processes $(S_t, t \geq 0)$ of index $\alpha \in (0,1)$ with $\mu' = \sigma^2 = 0$, $\nu(dx) = |x|^{-\alpha-1}(c_+ 1_{\{x>0\}} + c_- 1_{\{x<0\}})dx$ for further parameters $c_\pm \geq 0$. They are the only Lévy processes with $S_{at} = a^{1/\alpha} S_t$.
2. Stable processes $(S_t, t \geq 0)$ of index $\alpha \in (1,2)$ with $\mu'' = \sigma^2 = 0$, $\nu(dx) = |x|^{-\alpha-1}(c_+ 1_{\{x>0\}} + c_- 1_{\{x<0\}})dx$ for further parameters $c_\pm \geq 0$. They are the only Lévy processes with $S_{at} = a^{1/\alpha} S_t$. Slight variation for $\alpha = 1$. Brownian motion $\alpha = 2$.
3. CGMY process $(X_t, t \geq 0)$ with $\sigma^2 = \mu'' = 0$ and $\nu(dx) = g(x)dx$

$$g(x) = \begin{cases} C_+ \exp\{-G|x|\}|x|^{-Y-1} & x > 0 \\ C_- \exp\{-M|x|\}|x|^{-Y-1} & x < 0 \end{cases}$$

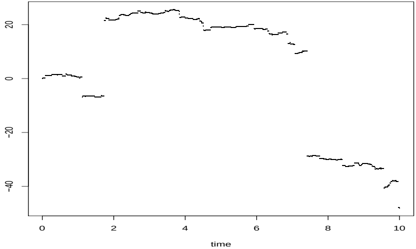
for five parameters $C_\pm > 0$, $G > 0$, $M > 0$, $Y \in [0,2)$.

Advantages: more modelling freedom, can be easily generalised further, since any measure ν is admissible subject to $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$.
Disadvantages: probability density function not available explicitly

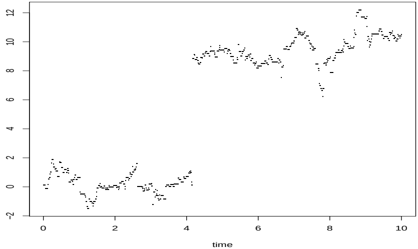
Symmetric stable processes for parameters $\alpha \in \{0.5, 1, 1.5, 1.8\}$.



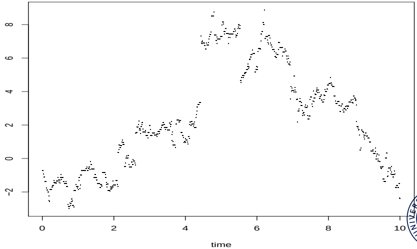
Stable process with index 0.5 and cplus= 1 and cminus= 1



Stable process with index 1 and cplus= 1 and cminus= 1

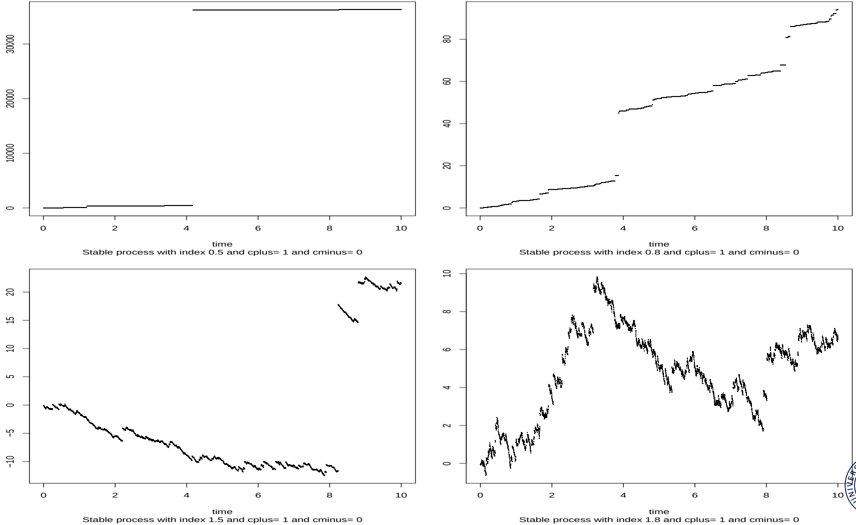


Stable process with index 1.5 and cplus= 1 and cminus= 1



Stable process with index 1.8 and cplus= 1 and cminus= 1

Stable processes with no negative jumps for $\alpha \in \{0.5, 0.8, 1.5, 1.8\}$.



Modelling and inference

Lévy processes are semimartingales. As a consequence, stochastic integrals

$$\int_0^t f(s)dX_s \qquad \text{and} \qquad \int_0^t Q(s)dX_s$$

are well-defined for L^2-functions f and predictable processes Q that are appropriately L^2 also.

Example ((Non-Gaussian) Ornstein-Uhlenbeck process)

Let $(X_t, t \geq 0)$ be a Lévy process and $\lambda \in (0, \infty)$. Then we call

$$Y_t = e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)} dX_s, \qquad t \geq 0,$$

the associated Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process is a Markov process and possesses a stationary regime under a mild log-moment condition.

Further classes of Lévy-driven stochastic processes can be obtained

- by stochastic differential equations

$$dY_t = F(Y_t)dt + G(Y_{t-})dX_t.$$

  For $F(y) = -\lambda y$ and $G(y) \equiv 1$ this is again the Ornstein-Uhlenbeck process.

- by time-change/subordination

$$X_{u(s)} \qquad \text{or} \qquad X_{\tau(s)}$$

  e.g. for a deterministic increasing function u, or for an increasing stochastic process τ, either independent of X or such that $\tau(s)$ is a stopping time for all s.

Modelling financial price processes

Definition (Black-Scholes model)

The Black Scholes model for a financial price process is the solution to the stochastic differential equation $dS_t = S_t\mu dt + S_t\sigma dW_t$, $S_0 > 0$, which is solved by

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}.$$

There is abundant empirical evidence that financial price processes do not have Normally distributed returns $\log(S_{t+s} - S_t)$.

Definition (Lévy market)

In a Lévy market, the price process is taken as $S_t = S_0 \exp\{X_t\}$ for a Lévy process $(X_t, t \geq 0)$.

Stochastic volatility models

Lévy markets give good fit to return distributions, but there is also evidence against independence of increments. Many stochastic volatility models have been considered.

E.g., Barndorff-Nielsen and Shephard introduced the following Lévy-driven stochastic volatility model:

$$dY_t = \sigma_{t-} dW_t, \quad d\sigma_t = -\lambda \sigma_t dt + dZ_t,$$

where Z is an increasing Lévy process and W an independent Brownian motion.

- The price process $(Y_t, t \geq 0)$ has continuous sample paths.
- This means that the volatility process $(\sigma_t, t \geq 0)$ is a non-Gaussian Ornstein-Uhlenbeck process.
- Interpretation: upward jumps correspond to new information leading to increased activity that then slows down.
- The volatility process $(\sigma_t, t \geq 0)$ is not (directly) observable.

Quadratic variation and realised power variation

Let $Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s$ be a (continuous) Brownian semimartingale (a stochastic process with stochastic volatility $(\sigma_s, s \geq 0)$).

To make inference on to unobserved volatility process from the observed price process, consider

$$\textbf{CiP} \quad \{Y_\delta\}_t^{(m)} = \sum_{k=0}^{[t/\delta]-m} \prod_{k=1}^{m} (Y_{(j+k)\delta} - Y_{(j+k-1)2^{-n}})^{2/m} \to d_m \int_0^t \sigma_s^2 ds = d_m [Y]_t$$

in probability as $\delta \downarrow 0$. The left-hand side is called realised m-power variation (or realised variation for $m = 1$), while the right-hand side is called quadratic variation. This convergence holds under very general assumptions. Under slightly more restrictive assumptions,

$$\textbf{CLT} \quad \delta^{-1/2} \left(\{Y_\delta\}_t^{(m)} - d_m [Y]_t \right) \to c_m \int_0^t \sigma_s^2 d\beta_s$$

in distribution for an independent Brownian motion β.

Application: Volatility inference with jumps

Suppose now that the price process includes a discontinuous component $S_t = Y_t + X_t$ where

- $Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s$ is a Brownian semimartingale
- and $(X_t, t \geq 0)$ is a Lévy process without Brownian term, not necessarily independent of Y.
- Define $\alpha = \inf\{\gamma \geq 0 : \int_{[-1,1]} |x|^\gamma \nu(dx) < \infty\} \in [0, 2]$.
- We are still interested in making inference on $(\sigma_t, t \geq 0)$.

Theorem (Inference in the presence of jumps)

1. *If $\alpha < 2$ and $m \geq 2$, then CiP is valid with Y replaced by S.*
2. *If $\alpha < 1$ and $\alpha/(2-\alpha) < 2/m < 1$, then CLT is valid with Y replaced by S.*

1. Applebaum, D.: *Lévy processes and stochastic calculus.* CUP 2004
2. Asmussen, S. and Rosinski, J.: Approximations of small jumps of Lévy processes with a view towards simulation. *J. Appl. Prob.* **38** (2001), 482–493
3. Barndorff-Nielsen, O.E., Shephard, N.: Realised power variation and stochastic volatility. *Bernoulli* **9** (2003), 243–265. Correction published in pages 1109–1111
4. Barndorff-Nielsen, O.E., Shephard, N. and Winkel, M.: Limit theorems for multipower variation in the presence of jumps. *Stoch. Proc. Appl.* **116** (2006), 796–806
5. Bertoin, J.: *Lévy processes.* CUP 1996
6. Kyprianou, A.E.: *Introductory lectures on fluctuations of Lévy processes with applications.* Springer 2006
7. Sato, K.: *Lévy processes and infinitely divisible distributions.* CUP 1999
8. Schoutens, W.: *Lévy processes in finance.* Wiley 2001

R code for some of the simulations

```
psum <- function(vector){
  b=vector;
  b[1]=vector[1];
  for (j in 2:length(vector)) b[j]=b[j-1]+vector[j];
  b}
gammarw <- function(a,p){
  unif=runif(10*p,0,1)
  pos=qgamma(unif,a/p,a);
  space=psum(pos);
  time=(1/p)*1:(10*p);
  plot(time,space,
    pch=".",
    sub=paste("Gamma process with shape parameter",a,
              "and scale parameter",a))}
```

```
vgammarw <- function(a,p){
  unifpos=runif(10*p,0,1)
  unifneg=runif(10*p,0,1)
  pos=qgamma(unifpos,a*a/(2*p),a);
  neg=qgamma(unifneg,a*a/(2*p),a);
  space=psum(pos-neg);
  time=(1/p)*1:(10*p);
  plot(time,space,
    pch=".",
    sub=paste("Variance Gamma process with shape
           parameter",a*a/2,"and scale parameter",a))}
```

Simulations can now been carried out with various values of parameters $a > 0$ and steps per time unit $p = 1/\delta$ in `gammarw(a,p)`, e.g.

```
gammarw(10,100)
vgammarw(10,1000)
```

```
stableonesided <- function(a,c,eps,p){
  f=c*eps^(-a)/a;
  n=rpois(1,10*f);t=runif(n,0,10);
  y=(eps^(-a)-a*f*runif(n,0,1)/c)^(-1/a);
  ytemp=1:n;res=(1:(10*p))/100;{
  for (k in 1:(10*p)){{for (j in 1:n){
      if(t[j]<=k/p)ytemp[j]<-y[j] else ytemp[j]<-0}};
    res[k]<-sum(ytemp)}};
  res}
stable <- function(a,cp,cn,eps,p){
  pos=stableonesided(a,cp,eps,p);
  neg=stableonesided(a,cn,eps,p);
  space=pos-neg;time=(1/p)*1:(10*p);
  plot(time,space,pch=".",
    sub=paste("Stable process with index",a,
              "and cplus=",cp,"and cminus=",cn))}
```

```
stableonesidedcomp <- function(a,c,eps,p){
  f=(c*eps^(-a))/a;
  n=rpois(1,10*f);
  t=runif(n,0,10);
  y=(eps^(-a)-a*f*runif(n,0,1)/c)^(-1/a);
  ytemp=1:n;
  res=(1:(10*p))/100;{
  for (k in 1:(10*p)){{if (n!=0)for (j in 1:n){
    if (t[j]<=k/p)ytemp[j]<-y[j] else ytemp[j]<-0}};{
    if (n!=0)res[k]<-sum(ytemp)-(c*k/(p*(a-1)))*(eps^(1-a))
      else res[k]<--c*k/(p*(a-1))*(eps^(1-a))}}};
  res}
```

This R code to simulate stable processes can be refined to more general measures ν with smooth densities using the rejection method.