suggested solutions to practice problems

You must show necessary details in order to get full points.

1. Calculate the following limit, if it exists.

   (a) $\lim_{x\to 0} \dfrac{\sin^{-1} x}{\sin x}$

   Sol: Note that $\lim_{x\to 0} \sin x = 0$, $\lim_{x\to 0} \sin^{-1} x = 0$, therefore the limit is of the form $\frac{0}{0}$. Clearly, both numerator and denominator are differentiable. So we apply L'Hospital's rule

   $$\lim_{x\to 0} \frac{\sin^{-1} x}{\sin x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x} = \lim_{x\to 1} \frac{1}{\cos x\sqrt{1-x^2}} = 1.$$

   (b) $\lim_{x\to\infty} \dfrac{\ln x^2}{x^{\frac{1}{3}}}$

   Sol: Note that $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^{\frac{1}{3}} = \infty$. Therefore, L'Hospital's rule gives

   $$\lim_{x\to\infty} \frac{\ln x^2}{x^{\frac{1}{3}}} = \lim_{x\to\infty} \frac{2\ln x}{x^{\frac{1}{3}}} = \lim_{x\to\infty} \frac{\frac{2}{x}}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x\to\infty} \frac{6}{x^{\frac{1}{3}}} = 0.$$

   (c) $\lim_{x\to 0} (1 - 2(x^2 + x))^{\frac{5}{x^3 - x^2}}$

   left as exercise. (the limit does not exist).

2. Find y' if $-e^{y^2} + \tan xy = 0$.

   Sol: By viewing $y = y(x)$ and differentiating both sides with respect to x using chain rule yields

   $$-e^{y^2} 2yy' + \sec^2 xy(y + xy') = 0.$$

   Rearranging the terms give

   $$y' = \frac{y \sec^2 xy}{2ye^{y^2} - x \sec^2 xy}.$$

3. Let $f''(x) = 2\sin 2t + 3$, $f'(0) = 1$, $f(\pi) = 3$. Find $f(x)$.

   Sol: The antiderivative of f'' is f', and in general form it is given by

   $$f' = -\cos 2t + 3t + C,$$

where we have used the relation $\frac{d}{dt}\cos 2t = -2\sin 2t$. The condition $f'(0) = 1$ gives

$$-\cos 0 + 3 \times 0 + C = 1,$$

from which we get $C = 2$. Hence $f' = -\cos 2t + 3t + 2$. Now the antiderivative of f' is f, and in general form it is given by

$$f = -\frac{1}{2}\sin 2t + \frac{3}{2}t^2 + 2t + C.$$

The condition $f(\pi) = 3$ gives

$$-\frac{1}{2}\sin 2\pi + \frac{3}{2}\pi^2 + 2\pi + C = 3,$$

i.e., $C = 3 - \frac{3}{2}\pi^2 - 2\pi$. Therefore, $f(t) = -\frac{1}{2}\sin 2t + \frac{3}{2}(t^2 - \pi^2) + 2(t - \pi) + 3$.

4. Find the domain, range and derivative of the function $f(x) = \ln \cos^{-1} x$.

   Sol: According to the definition of $\ln x$ and $\cos^{-1} x$, x satisfies

   $$-1 \leq x \leq 1, \cos^{-1} x > 0$$

   Combining these two identities gives the domain $-1 \leq x < 1$, i.e., $[-1, 1)$. The function $\cos^{-1} x$ maps the domain $[-1, 1)$ to $(0, \pi]$, and $\ln x$ maps $(0, \pi]$ to $(-\infty, \ln \pi]$. Hence the range of the function is $(-\infty, \ln \pi]$.

   The derivative is given by

   $$f' = \frac{1}{\cos^{-1} x}(\cos^{-1} x)' = \frac{1}{\cos^{-1} x}\frac{-1}{\sqrt{1 - x^2}} = \frac{-1}{\sqrt{1 - x^2}\cos^{-1} x}$$

5. Let $f(x) = \frac{1+x^2}{1-x^2}$.

   (a) Find all critical numbers of f.

   Sol:
   $$f'(x) = \frac{2x(1 - x^2) - (1 + x^2)(-2x)}{(1 - x^2)^2} = \frac{4x}{(1 - x^2)^2}$$
   Hence, the critical numbers are $x = -1, 0, 1$.

   (b) Find the intervals where f is increasing and decreasing.

   Sol: It follows from part (a) that f is decreasing on $(-\infty, -1) \cup (-1, 0)$, and increasing on $(0, 1) \cup (1, \infty)$.

(c) Find the intervals where f is concave upward and concave downward.

Sol:

$$f'' = \frac{4}{(1-x^2)^2} + (-2)\frac{4x}{(1-x^2)^3}(-2x) = \frac{4(1-x^2)+16x^2}{(1-x^2)^3} = \frac{4(1+3x^2)}{(1-x^2)^3}$$

Hence the function f is concave upward on $(-1,1)$ and concave downward on $(-\infty,-1)\cup(1,\infty)$.

(d) Find the local maximum and minimum.

Sol: At the critical numbers $x = \pm 1$, f' does not change sign. At $x = 0$, f' changes from negative to positive, and hence $x = 0$ is a point of local minimum.

6. Find the area of the largest rectangle that can be inscribed inside a circle of radius 1 cm.

Sol: Without loss of generality, put the center of the circle at the origin. Let the vertex of the rectangle in the first quadrant be (x, y), with $x, y \geq 0$. Then we have $x^2 + y^2 = 1$. The two sides of the rectangle have length $2x$ and $2y$, respectively, due to the symmetry. Therefore, the area of the rectangle is $2x \times 2y = 4xy$.

Solving the first identity gives $y = \sqrt{1-x^2}$. Upon substituting y into $4xy$, we get the area function $f(x) = 4x\sqrt{1-x^2}$, $0 \leq x \leq 1$. To maximize f, we compute the critical number:

$$f' = 4\sqrt{1-x^2} - 4x\frac{x}{\sqrt{1-x^2}} = 4\frac{1-2x^2}{\sqrt{1-x^2}}.$$

Therefore, there are two critical numbers $x = \frac{\sqrt{2}}{2}$ and $x = 1$. ($x = -1, -\frac{\sqrt{2}}{2}$ are outside the domain). Further, we have $f(\frac{\sqrt{2}}{2}) = 4\frac{\sqrt{2}}{2} \times \sqrt{1-(\frac{\sqrt{2}}{2})^2} = 2$, $f(1) = 0$, $f(0) = 1$ (end point). Therefore, $x = \frac{\sqrt{2}}{2}$ is indeed the abs. max.. The corresponding rectangle is a square of width $\sqrt{2}$.

7. Use the fundamental theorem of calculus in the following problems.

(a) Let the function $f(x)$ be defined by $f(x) = \displaystyle\int_{\sqrt{x}}^{0} (\tan t + t^2)dt$. Find $f'(x)$.

Sol: Let $u = \sqrt{x}$. Then by chain rule we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_{\sqrt{x}}^{0}(\tan t + t^2)dt = \frac{d}{dx}(-\int_{0}^{\sqrt{x}}(\tan t + t^2)dt) = -\frac{d}{du}\int_0^u (\tan t + t^2)\cdot\frac{du}{dx} \\ &= -(\tan u + u^2)\frac{d}{dx}\sqrt{x} \\ &= -\frac{1}{2}x^{-\frac{1}{2}}(\tan\sqrt{x} + x). \end{aligned}$$

(b) Compute the definite integral $\int_{-2}^{2}(x^2 - x + \cos x)dx$.

Sol: One antiderivative of $x^2 - x + \cos x$ is $\frac{x^3}{3} - \frac{x^2}{2} + \sin x$. Therefore, by the fundamental theorem of calculus, we have

$$\begin{aligned}\int_{-2}^{2}(x^2 - x + \cos x)dx &= \frac{x^3}{3} - \frac{x^2}{2} + \sin x\Big|_{-2}^{2} \\ &= (\frac{2^3}{3} - \frac{2^2}{2} + \sin 2) - (\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + \sin(-2)) = \frac{16}{3} + 2\sin 2\end{aligned}$$

8. Find the values of c and d that make h continuous on $\mathbb{R}$:

$$h(x) = \begin{cases} 2x & \text{if } x < 1, \\ cx^2 + d & \text{if } 1 \le x \le 2, \\ 4x & \text{if } x > 2. \end{cases}$$

Sol: The function clearly is continuous everywhere except $x = 1, 2$. It suffices to make it continuous at these two points. To this end, it is enough to require

$$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^-} h(x), \quad \lim_{x\to 2^+} h(x) = \lim_{x\to 2^-} h(x)$$

However,

$$\begin{aligned}\lim_{x\to 1^+} h(x) &= c1^2 + d = c + d, \\ \lim_{x\to 1^-} h(x) &= 2 \times 1 = 2, \\ \lim_{x\to 2^+} h(x) &= 4 \times 2 = 8, \\ \lim_{x\to 2^-} h(x) &= c2^2 + d = 4c + d.\end{aligned}$$

This gives a system

$$c + d = 2, \quad 4c + d = 8$$

solving the system gives $c = 2$, $d = 0$.