High-order extrapolation

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In many fields of computational physics, it is often required to extrapolate a function from a region where it is known to a region where it is unknown. In his 2003 paper [1], T. Aslam presents a general methodology for multidimensional extrapolation. In this approach, the polynomial extrapolation can be formulated by solving a series of linear partial differential equations, which results in a more robust and flexible method than the geometric scheme commonly used (i.e. closest point methods).

As an example, we'll study the case of constant extrapolation. Let there be a function u which is defined only in a portion of space, and we would like to extrapolate it into the remaining areas of space (as in figure 1). We assume there exists a level set function ψ such that $\psi \leq 0$ defines the region where u is known, and $\psi > 0$ is the region where u needs to be extrapolated. ψ is typically the signed distance function from the interface Γ.

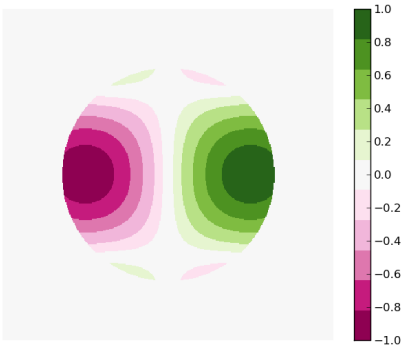

Figure 1 *Initial conditions for Aslam's extrapolation scheme. The function is defined only in the inner region of radius 2, as* $u(x, y) = cos(x)sin(y)$.

In the case of constant extrapolation, the value of u at the interface Γ is extrapolated as a constant value along the normal direction (as seen in figure 2),

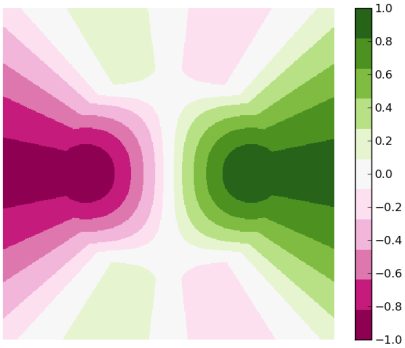

Figure 2 *Constant extrapolation in 400-cells grid.*

which is defined everywhere in space as

$$\hat{n} = \frac{\vec{\nabla}\psi}{\left|\vec{\nabla}\psi\right|}$$

The PDE used to achieve constant extrapolation is

$$\frac{\partial u}{\partial t} + H(\psi)\hat{n} \cdot \vec{\nabla} u = 0 \tag{1}$$

where $H(\psi)$ is the Heaviside function

$$H(\psi) = \begin{cases} 1 & \text{if } \psi > 0 \\ 0 & \text{if } \psi \leq 0 \end{cases}$$

So, to achieve constant extrapolation, one can simply solve equation 1 to steady state. Once the PDE is steady, then we have $\hat{n} \cdot \vec{\nabla} u = 0$, which yields that u will be constant along the characteristic direction $\hat{n}$.

In his paper, T. Aslam to extend this method to higher-order polynomial methods, up to quadratic extrapolation (as seen in figures 3 and 4).

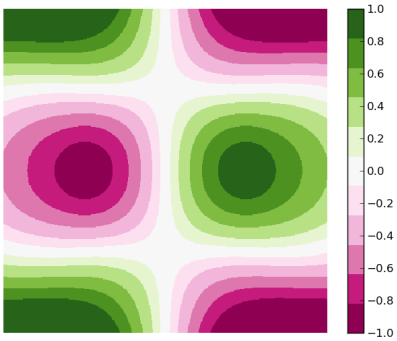

Figure 3 *Linear extrapolation in 400-cells grid.*

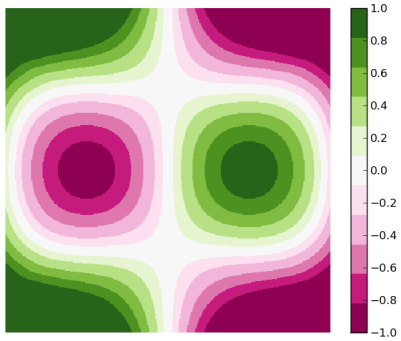

Figure 4 *Quadratic extrapolation in 400-cells grid.*

In tables 1–3 we show the convergence rates for our implementations of constant, linear and quadratic extrapolations:

- R_1: rate for the error in the L_1 norm, taken over the entire grid.
- R_∞: rate for the error in the L_∞ norm, taken over the entire grid.
- $R_{avg-band}$: rate for the mean error in the L_1 norm, taken over a 3-cells band around the interface.
- $R_{\infty-band}$: rate for the error in the L_∞ norm, taken over a 3-cells band around the interface.

n	R_1	R_∞	$R_{avg-band}$	$R_{\infty-band}$
100				
200	0.77	0.86	0.98	0.93
400	0.98	1.05	1.09	0.89
800	0.99	0.85	1.05	1.05

Table 1 *Numerical accuracy for constant extrapolation.*

n	R_1	R_∞	$R_{avg-band}$	$R_{\infty-band}$
100				
200	0.73	0.91	2.10	1.75
400	0.86	0.69	2.02	1.87
800	0.97	1.02	2.08	1.98

Table 2 *Numerical accuracy for linear extrapolation.*

n	R_1	R_∞	$R_{avg-band}$	$R_{\infty-band}$
100				
200	1.02	0.98	2.55	2.53
400	1.07	1.02	2.33	2.41
800	1.02	1.07	2.10	2.12

Table 3 *Numerical accuracy for quadratic extrapolation.*

Even though globally only first-order results are achieved, it is usually more important to get higher convergence rates in the vicinity of the interface Γ.

References

[1] T. D. ASLAM, *A partial differential equation approach to multidimensional extrapolation*, **J. Comp. Phys.** 193 (2004), 349-355.