Quasi-classical Logic: Non-trivializable classical reasoning from inconsistent information

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Abstract. Here we present a new paraconsistent logic, called quasi-classical logic (or QC logic) that allows the derivation of non-trivializable classical inferences. For this it is necessary that queries are in conjunctive normal form and the reasoning process is essentially that of clause finding. We present a proof-theoretic definition, and semantics, and show that the consequence relation observes reflexivity, monotonicity and transitivity, but fails cut and supraclassicality. Finally we discuss some of the advantages of this logic, over other paraconsistent logics, for applications in information systems.

1 Introduction

In practical reasoning, it is common to have 'too much' information about some situation. In other words, it is common to have to reason with classically inconsistent information. The diversity of logics proposed for aspects of practical reasoning indicates the complexity of this form of reasoning. However, central to practical reasoning seems to be the need to reason with inconsistent information without the logic being trivialized (Gabbay 1991, Finkelstein 1994). This is the need to derive reasonable inferences without deriving the trivial inferences that follow from ex falso quodlibet (EFQ):

$$\frac{\bot}{\alpha}$$

So for example, from the data $\{\alpha, \neg\alpha, \alpha \rightarrow \beta, \delta\}$, reasonable inferences include α, $\neg\alpha$, $\alpha \rightarrow \beta$, and δ by reflexivity, β by modus ponens, $\alpha \wedge \beta$ by introduction of conjunction, and $\neg\beta \rightarrow \neg\alpha$ by contraposition. In contrast, trivial inferences include γ, $\gamma \wedge \neg\delta$, etc, by ex falso quodlibet.

For classical logic, EFQ means that any conclusion can be drawn from inconsistent information. This renders the information useless, and therefore classical logic is obviously unsatisfactory for handling it. A possible solution is to weaken classical logic by dropping reductio ad absurdum. This gives a class of logics called paraconsistent logics such as C_ω (da Costa 1974). However, the weakening of the proof rules means that the connectives in the language do not behave in a classical fashion (Besnard 1991). For example, disjunctive syllogism does not hold, $((\alpha \vee \beta) \wedge \neg\beta) \rightarrow \alpha$ whereas modus ponens does hold. So, for example, α does not follow from $\{(\alpha \vee \beta), \neg\beta\}$, whereas α does follow from $\{(\neg\beta \rightarrow \alpha), \neg\beta\}$.

There are many similar examples that could be confusing and counter-intuitive for users of such a practical reasoning system.

An alternative, giving quasi-classical logic (or QC logic), which we explore in this paper, is to restrict the queries to being in conjunctive normal form, and restrict the proof theory to that of finding clauses that follow from the data. In the following we present a proof theory, and semantics for QC, and show how it provides a useful form of reasoning from inconsistent information.

2 Language for QC

In the following we present the usual classical definition for the language. In addition, we define the notion of a clause being trivial with respect to a set of formulae.

Definition 2.1 *Let $\mathcal{L}$ be the set of classical propositional formulae formed from a set of atoms and the $\wedge, \vee$ and $\neg$ connectives.*

Definition 2.2 *For each atom $\alpha \in \mathcal{L}$, α is a literal and $\neg\alpha$ is a literal. For $\alpha_1 \vee .. \vee \alpha_n \in \mathcal{L}$, $\alpha_1 \vee .. \vee \alpha_n$ is a clause iff each of $\alpha_1, .., \alpha_n$ is a literal. For $\alpha_1 \wedge .. \wedge \alpha_n \in \mathcal{L}$, $\alpha_1 \wedge .. \wedge \alpha_n$ is in a conjunctive normal form (CNF) iff each of $\alpha_1, .., \alpha_n$ is a clause.*

Definition 2.3 *For $\alpha_1 \wedge .. \wedge \alpha_n \in \mathcal{L}$, and $\beta \in \mathcal{L}$, $\alpha_1 \wedge .. \wedge \alpha_n$ is in a conjunctive normal form (CNF) of β iff $\alpha_1, .., \alpha_n$ is classically equivalent to β, and $\alpha_1, .., \alpha_n$ is in a CNF.*

For any $\alpha \in \mathcal{L}$, a CNF of α can be produced by the application of distributivity, double negation elimination, and de Morgan laws. We require the following function $Atoms(\Delta)$ which gives the set of atoms used in the set of formulae in Δ.

Definition 2.4 *Let $\Delta \in \wp(\mathcal{L})$, and α, β, $\gamma_1 \wedge .. \wedge \gamma_n$, $\delta_1 \vee .. \vee \delta_n \in \mathcal{L}$,*

$$Atoms(\Delta \cup \{\beta\}) = Atoms(\{\beta\}) \cup Atoms(\Delta)$$

$$Atoms(\emptyset) = \emptyset$$

$$Atoms(\{\beta\}) = Atoms(\{\gamma\}) \text{ where } \gamma \text{ is the CNF of } \beta$$

$$Atoms(\{\gamma_1 \wedge .. \wedge \gamma_n\}) = Atoms(\{\gamma_1\}) \cup .. \cup Atoms(\{\gamma_n\})$$

$$Atoms(\{\delta_1 \vee .. \vee \delta_n\}) = Atoms(\{\delta_1\}) \cup .. \cup Atoms(\{\delta_n\})$$

$$Atoms(\{\neg\alpha\}) = Atoms(\{\alpha\})$$

$$Atoms(\{\alpha\}) = \{\alpha\} \text{ if } \alpha \text{ is an atom}$$

Definition 2.5 *A clause $\alpha \in \mathcal{L}$ is trivial with respect to Δ iff $Atoms(\Delta) \cap Atoms(\{\alpha\}) = \emptyset$.*

3 Proof theory for QC

In the following, we present the QC proof rules, which are a subset of the classical proof rules, and we define the notion of a QC proof, which is a restricted version of a classical proof.

3.1 The QC proof rules

Definition 3.1 *Assume that $\wedge$ is a commutative and associative operator, and $\vee$ is a commutative and associative operator.*

$$\frac{\alpha \wedge \beta}{\alpha} \text{ [Conjunct elimination] } \quad \frac{\alpha \vee \alpha \vee \beta}{\alpha \vee \beta} \text{ [Disjunct contraction]}$$

$$\frac{\alpha}{\neg\neg\alpha} \text{ [Negation introduction] } \quad \frac{\neg\neg\alpha}{\alpha} \text{ [Negation elimination]}$$

$$\frac{\alpha \vee \beta \qquad \neg\alpha \vee \gamma}{\beta \vee \gamma} \text{ [Resolution]}$$

$$\frac{\alpha \vee (\beta \wedge \gamma)}{(\alpha \vee \beta) \wedge (\alpha \vee \gamma)} \quad \frac{(\alpha \vee \beta) \wedge (\alpha \vee \gamma)}{\alpha \vee (\beta \wedge \gamma)} \text{ [Disjunct distribution]}$$

$$\frac{\alpha \wedge (\beta \vee \gamma)}{(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)} \quad \frac{(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\alpha \wedge (\beta \vee \gamma)} \text{ [Conjunct distribution]}$$

$$\frac{\neg(\alpha \wedge \beta)}{\neg\alpha \vee \neg\beta} \quad \frac{\neg\alpha \vee \neg\beta}{\neg(\alpha \wedge \beta)} \quad \text{[de Morgan laws]}$$

$$\frac{\neg\alpha \wedge \neg\beta}{\neg(\alpha \vee \beta)} \quad \frac{\neg(\alpha \vee \beta)}{\neg\alpha \wedge \neg\beta}$$

$$\frac{\alpha}{\alpha \vee \beta} \text{ [Disjunct introduction]}$$

3.2 Proofs in QC

Definition 3.2 *T is a proof-tree iff T is a tree where (1) each node is an element of $\mathcal{L}$; (2) for the trees with more than one node, the root is derived by application of any QC proof rule, where the premises for the proof rule are the parents of the root; (3) the leaves are the assumptions for the root; and (4) any node, that is not a leaf or root, is derived by the application of any QC proof rule - except the disjunct introduction rule - and the premises for the proof rule are the parents of the node.*

Definition 3.3 *Let $\Delta \in \wp(\mathcal{L})$. For a clause β, there is a QC proof of β from Δ iff there is a QC proof tree, where each leaf is an element of Δ, and the root is β.*

Definition 3.4 *Let $\Delta \in \wp(\mathcal{L})$, and $\alpha \in \mathcal{L}$. We define the QC consequence relation, denoted $\vdash_Q$, as follows:*

$$\Delta \vdash_Q \alpha \text{ iff for each } \beta_i \ (1 \leq i \leq n) \text{ there is a QC proof of } \beta_i \text{ from } \Delta$$

where $\beta_1 \wedge .. \wedge \beta_n$ is a CNF of α.

Examples 3.1 *For $\Delta = \{\alpha \vee \beta, \alpha \vee \neg\beta, \neg\alpha \wedge \delta\}$, consequences of Δ include $\alpha \vee \beta$, $\alpha \vee \neg\beta$, α, $\neg\alpha$, and δ, but do not include $\neg\delta$, γ, $\gamma \vee \phi$, or $\neg\psi \wedge \neg\phi$. For $\Delta = \{\alpha \vee (\beta \wedge \gamma), \neg\beta\}$, consequences of Δ include $\alpha \vee \beta$, $\alpha \vee \gamma$, α, and $\neg\beta$.*

3.3 Properties of the QC consequence relation

Proposition 3.1 *Let $\Delta \in \wp(\mathcal{L})$, and $\alpha \in \mathcal{L}$. If $\Delta \vdash_Q \alpha$, then α is not trivial with respect to Δ. In other words, $Atoms(\Delta) \cap Atoms(\{\alpha\}) \neq \emptyset$.*

Proposition 3.2 *Cut, defined as follows, fails for the QC consequence relation.*

$$\frac{\Delta \cup \{\alpha\} \vdash_Q \beta \qquad \Gamma \vdash_Q \alpha}{\Delta \cup \Gamma \vdash_Q \beta}$$

Proof Consider that $\{\neg\beta \vee \delta, \alpha\} \cup \{\neg\alpha \vee \beta\} \vdash_Q \delta$ and $\{\neg\alpha\} \vdash_Q \neg\alpha \vee \beta$, but that $\{\neg\beta \vee \delta, \alpha\} \cup \{\neg\alpha\} \not\vdash_Q \delta$

Proposition 3.3 *Unit cumulativity, defined as follows, fails for the QC consequence relation.*

$$\frac{\Delta \vdash_Q \beta \qquad \Delta \cup \{\beta\} \vdash_Q \gamma}{\Delta \vdash_Q \gamma}$$

Proof Consider that $\{\neg\alpha, \alpha\} \vdash_Q \alpha \vee \beta$ and $\{\neg\alpha, \alpha\} \cup \{\alpha \vee \beta\} \vdash_Q \beta$, but $\{\neg\alpha, \alpha\} \not\vdash_Q \beta$

Proposition 3.4 *Reflexivity, defined as follows, succeeds for the QC consequence relation.*

$$\Delta \cup \{\alpha\} \vdash_Q \alpha$$

Proposition 3.5 *Monotonicity, defined as follows, succeeds for the QC consequence relation.*

$$\frac{\Delta \vdash_Q \alpha}{\Delta \cup \{\beta\} \vdash_Q \alpha}$$

Proposition 3.6 *Supraclassicality, defined as follows, fails for the QC consequence relation.*

$$\Delta \vdash \alpha \text{ implies } \Delta \vdash_Q \alpha$$

Proposition 3.7 *For $\Delta = \emptyset$, there are no $\alpha \in \mathcal{L}$ such that $\Delta \vdash_Q \alpha$*

4 Semantics for QC

Definition 4.1 *Let S be some set. Let O be a set of objects defined as follows, where $+\alpha$ is a positive object, and $-\alpha$ is a negative object.*

$$O = \{+\alpha \mid \alpha \in S\} \cup \{-\alpha \mid \alpha \in S\}$$

We call any $X \in \wp(O)$ a model.

We can consider the following meaning for positive and negative objects being in or out of some model X,

- $+\alpha \in X$ means α is "satisfiable" in the model
- $-\alpha \in X$ means $\neg\alpha$ is "satisfiable" in the model
- $+\alpha \notin X$ means α is not "satisfiable" in the model
- $-\alpha \notin X$ means $\neg\alpha$ is not "satisfiable" in the model

This semantics can also be regarded as giving four truth values - called "Both", "True", "False", "Neither". For a literal α, and its complement α^*,

- α is "Both" if α is satisfiable and α^* is satisfiable
- α is "True" if α is satisfiable and α^* is not satisfiable
- α is "False" if α is not satisfiable and α^* is satisfiable
- α is "Neither" if α is not satisfiable and α^* is not satisfiable

This intuition coincides with that of four-valued logics (Belnap 1977). However, we will not follow the four-valued lattice-theoretic interpretation of the connectives, and instead provide a significantly different semantics.

Definition 4.2 *Let $\models_s$ be a satisfiability relation such that $\models_s \subseteq \wp(O) \times \mathcal{L}$. For $X \in \wp(O), \alpha \in \mathcal{L}$, we define $\models_s$ as follows,*

$$X \models_s \alpha \text{ if } +\alpha \in X$$

$$X \models_s \neg\alpha \text{ if } -\alpha \in X$$

$$X \models_s \alpha \wedge \beta \text{ iff } X \models_s \alpha \text{ and } X \models_s \beta$$

$$X \models_s \neg\neg\alpha \text{ iff } X \models_s \alpha$$

$$X \models_s \neg(\alpha \wedge \beta) \text{ iff } X \models_s \neg\alpha \vee \neg\beta$$

$$X \models_s \neg(\alpha \vee \beta) \text{ iff } X \models_s \neg\alpha \wedge \neg\beta$$

$$X \models_s \alpha \vee (\beta \wedge \gamma) \text{ iff } X \models_s (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$$

$$X \models_s \alpha \wedge (\beta \vee \gamma) \text{ iff } X \models_s (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

$$\begin{aligned} X \models_s \alpha \vee \beta \text{ iff } & [X \not\models_s \neg\alpha \text{ or } X \models_s \beta] \\ & \text{and } [X \not\models_s \neg\beta \text{ or } X \models_s \alpha] \\ & \text{and } [X \models_s \alpha \text{ or } X \models_s \beta] \end{aligned}$$

Definition 4.3 *We extend the notion of satisfaction to that of weak satisfaction, denoted as* $\models_w$*, as follows, where* $\models_w \subseteq \wp(O) \times \mathcal{L}$*:*

$$X \models_w \alpha \text{ if } X \models_s \alpha$$

$$X \models_w \alpha \vee \beta \text{ if } X \models_s \alpha$$

Definition 4.4 *Let* $\models_Q$ *be an entailment relation such that* $\models_Q \subseteq \wp(\mathcal{L}) \times \mathcal{L}$*, and defined as follows,*

$\{\alpha_1, .., \alpha_n\} \models_Q \beta$
iff for all models X *if* $X \models_s \alpha_1$ *and ... and* $X \models_s \alpha_n$ *then* $X \models_w \beta$

Examples 4.1 *Let* $\Delta = \{\alpha\}$*, and let* $X1 = \{+\alpha\}$ *and* $X2 = \{+\alpha, -\alpha\}$*. Now* $X1 \models_s \alpha$*, and* $X2 \models_s \alpha$*, whereas* $X1 \models_s \alpha \vee \beta$*, and* $X2 \not\models_s \alpha \vee \beta$*. However,* $X1 \models_w \alpha \vee \beta$*, and* $X2 \models_w \alpha \vee \beta$*, and indeed* $\Delta \models_Q \alpha \vee \beta$*. As another example, let* $\Delta = \{\alpha \vee \beta, \neg\alpha\}$*. For all models* X*, if* $X \models_s \alpha \vee \beta$*, and* $X \models_s \neg\alpha$*, then* $X \models_s \beta$*. Hence,* $\Delta \models_Q \alpha \vee \beta$*,* $\Delta \models_Q \neg\alpha$*, and* $\Delta \models_Q \beta$*.*

Proposition 4.1 *For all* $\alpha \in \mathcal{L}$*,* $\{\} \not\models_Q \alpha$*.*

Examples 4.2 *Consider* $\alpha \vee \neg\alpha$*. Here models that satisfy* $\{\}$ *include those, for example* X*, where* $+\alpha \notin X$ *and* $-\alpha \notin X$*, and so* $X \not\models_Q \alpha$ *and* $X \not\models_Q \neg\alpha$ *hold. Hence, it is not the case that for all models* X *that satisfies the empty set,* X *also satisfies* $\alpha \vee \neg\alpha$*.*

Proposition 4.2 *The* $\vdash_Q$ *relation is sound with respect to the* $\models_Q$ *relation.*

Proof See Besnard (1995).

Proposition 4.3 *The $\vdash_Q$ relation is complete with respect to the $\models_Q$ relation, where the formula on the right-hand side of both relations is in CNF.*

Proof See Besnard (1995).

5 Discussion

Developing a non-trivializable, or paraconsistent logic, necessitates some compromise, or weakening, of classical logic. The compromises imposed to give QC logic seem to be more appropriate than other paraconsistent logics for applications in computing. QC logic provides a means to obtain all the non-trivial resolvants from a set of formulae, without the problem of trivial clauses also following. Though the constraints on QC logic result in tautologies also being non-derivable, this is not usually a problem for applications.

QC logic exhibits the nice feature that no attention need to be paid to a special form that premises should have. This is in contrast with other paraconsistent logics where two formulae identical by definition of a connective in classical logic may not yield the same set of conclusions. An example given earlier in this paper is $\{(\neg\alpha \rightarrow \beta), \neg\alpha\}$ yielding the conclusion β as opposed to $\{\alpha \vee \beta, \neg\alpha\}$. QC logic is much better behaved in this respect, as illustrated by the the fact that more non-trivial classical conclusions are captured by QC-logic.

QC logic is more flexible than resolution (for the clause finding version, consult (Lee 1967)) which only deals with formulas in clausal form. In fact, QC logic could even be extended easily to handle arbitrary formulas, including implicative formulas. It only takes the following two inference rules to accommodate such formulae,

$$\frac{\alpha \rightarrow \beta}{\neg\alpha \vee \beta}$$

and

$$\frac{\neg(\alpha \rightarrow \beta)}{\alpha \wedge \neg\beta}$$

QC logic is also more appropriate than various approaches to reasoning from consistent subsets of inconsistent sets of formulae (for a review, see Benferhat 1993). In particular, QC logic does not suffer from the limitation due to "breaking off" formulae into compatible pieces: QC logic can make use of the contents of the formulas without being constrained by a consistency check. Moreover, it is obviously an advantage of QC logic to dispense with the costly consistency checks that are needed in all approaches to reasoning from consistent subsets. Finally, QC logic lends itself to a thorough analysis in terms of inference as proved in section 3.3 of this paper. In this way, we actually know what QC logic is, what it can do, what its limitations are, and, for these reasons, how it should be used.

In relation to other work, the proof theory we give for QC is quite close to Schutte's K1 axiomatization of classical logic (Schutte 1950). Also, QC logic extends the natural idea of tautological entailment without losing the demand that any conclusion is somehow contained in each disjunct of a conjunct when the premise is in a conjunctive normal form. In fact, our model theory for QC can be viewed as a natural simplification of Dunn's semantics for tautological entailement (Anderson 1975).

6 Acknowledgements

This work was funded by the ESPRIT DRUMS2 project, and by the UK Engineering and Physical Science Research Council project number GR/J 15483.

7 References

Anderson A R and Belnap N D Jr. (1975) Entailment: The logic of relevance and necessity, Princeton University Press

Belnap N (1977) A useful four-valued logic, in Dunn J and Epstein G, Modern Uses of Multiple-Valued Logic, 5-37, Reidel

Benferhat S, Dubois D, Prade H (1993) Argumentative inference in uncertain and inconsistent knowledge bases, Proceedings of the 9th Conference on Uncertainty in Artificial Intelligence, 411-419, Morgan Kaufmann

Besnard Ph (1991) Paraconsistent logic approach to knowledge representation, in de Glas M, and Gabbay D, Proceedings of the First World Conference on Fundamentals of Artificial Intelligence, Angkor

Besnard Ph and Hunter A (1995) Properties of quasi-classical logic, Technical Report, Department of Computing, Imperial College, London

da Costa N C (1974) On the theory of inconsistent formal systems, Notre Dame Journal of Formal Logic, 15, 497-510

Finkelstein A, Gabbay D, Hunter A, Kramer J, and Nuseibeh B (1993) Inconsistency handling in multi-perspective specifications, in IEEE Transactions on Software Engineering, 20(8), 569-578

Gabbay D and Hunter A (1991) Making inconsistency respectable, Part 1, in Jorrand Ph. and Keleman J, Fundamentals of Artificial Intelligence Research, Lecture Notes in Artificial Intelligence, 535, 19-32, Springer

Lee R C T (1967) A completeness theorem and a computer program for finding theorems derivable from given axioms. PhD dissertation, University of California, Berkeley

Schutte K (1950) Schlussweisen-Kalkuele der Praedikatenlogik, Mathematische Annalen, 122, 47-65

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