Argumentative logics: reasoning with classically inconsistent information

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Abstract

Classical logic has many appealing features for knowledge representation and reasoning. But unfortunately it is flawed when reasoning about inconsistent information, since anything follows from a classical inconsistency. This problem is addressed by introducing the notions of "argument" and of "acceptability" of an argument. These notions are used to introduce the concept of "argumentative structures". Each definition of acceptability selects a subset of the set of arguments, and an argumentative structure is a subset of the power set of arguments. In this paper, we consider, in detail, a particular argumentative structure, where each argument is defined as a classical inference together with the applied premisses. For such arguments, a variety of definitions of acceptability are provided, the properties of these definitions are explored, and their inter-relationship described. The definitions of acceptability induce a family of logics called argumentative logics which we explore. The relevance of this work is considered and put in a wider perspective.

1 Introduction

Classical logic has many appealing features for knowledge representation and reasoning, but unfortunately when reasoning with inconsistent information, i.e. drawing conclusions from a database, the set of classical consequences is trivialized. In classical proof theory, anything follows from an inconsistency, and in classical semantics, there is no model of inconsistent information.

There have been a number of proposals to address this problem, two of which are prominent. The first, which we will not consider in any detail, is the wide range of paraconsistent logics (Besnard 1991, da Costa 1974). Normally, a paraconsistent logic is obtained by modifying classical consequence to avoid trivialization. Standard paraconsistent logics are monotonic, adding further premisses do not invalidate deductions, but since contradictions are tolerated, new interpretations have to be invented for the classical connectives. The other approach of particular interest is that of classical reasoning with consistent subsets of the database. This allows some of the useful inferences from a database to be derived but does not allow the trivial inferences. When reasoning from consistent subsets, classic logic can be applied without restriction. But this kind of reasoning is non-monotonic, since adding further premisses may violate the consistency of the subset in the context of which deductions have been made. We prefer the approach of reasoning from consistent subsets, because the classical semantics of the connectives prevails.

In the literature on paraconsistent logics, several motivations of the need to reason from inconsistent information can be found. We see a need to be able to handle inconsistencies in intelligent information systems, because inconsistency may be introduced into some context through, for instance, the application of conflicting "rules of thumb". For us, the issue is not how inconsistency arises or whether it is real or imagined. The issue is, that for the purpose of reasoning, conflicting sources of information may lead to contradictions and trivialization is not a viable answer to such contradictions. Rather, the contradictions need to be controlled and inconsistent information used in the process of reasoning. However, inferences that follow from consistent subsets of an inconsistent database are only weakly justified in general. To handle this problem we introduce the notion of an argument from a database, and a notion of acceptability of an argument. An argument is a subset of the database, together with an inference from that subset. Using the notion of acceptability defined over the set of all such arguments, the set of all arguments can be partitioned into sets of (arguments of) different degrees of acceptability. We introduce the notion of an argumentative structure, and define a specific argumentative structure to be characterized by a set of arguments and a notion of acceptability.

It is apparent that a very wide range of argumentative structures are definable. In principle any consequence relation and any notion of acceptability can be applied to characterize an argumentative structure. In this paper we will investigate a specific argumentative structure called A^*. We consider A^* as having a distinguished position among the argumentative structures definable from classical logic, because it is based on an intuitive notion of acceptability. For each set of arguments of a certain acceptability we induce a consequence relation, and call the corresponding family of logics, argumentative logics. Even though some of these consequence relations have been proposed and studied by other authors, we collate and extend these results.

We consider it to be of importance to understand basic argumentative structures, like A^*, before proposing more exotic ones, based on more advanced consequence relations than classical logic and other notions of acceptability. Throughout the main body of this paper we will concentrate on the argumentative structure A^*.

2 Basic definitions and results

We apply a standard notation throughout and use the same notation for meta and object formulas when this does not cause unnecessary confusion.

Definition 2.1 $\vdash$ *is classical entailment, defined in the standard way over the usual classical language (finite or countably infinite)* $\mathcal{L}$.

In particular classical entailment enjoys the properties of reflexivity, monotonicity, cut and compactness, in addition to the standard properties of the logical connectives.

Definition 2.2 *A database,* Δ*, is a set of sentences in* $\mathcal{L}$.

Definition 2.3 *For a database* Δ*,* $\mathsf{Cn}(\Delta)$ *is the set* $\{\phi|\Delta \vdash \phi\}$.

Definition 2.4 (Arguments) *An argument from* Δ *is a pair,* (Π, ϕ)*, such that* $\Pi \subseteq \Delta$ *and* $\Pi \vdash \phi$*. An argument is consistent, if* Π *is consistent. For a database* Δ*, we denote the set of arguments from* Δ *as* $\mathsf{An}(\Delta)$*, where* $\mathsf{An}(\Delta) = \{(\Pi, \phi)|\Pi \subseteq \Delta \wedge \Pi \vdash \phi\}$*.* Γ *is an argument set of* Δ *iff* $\Gamma \subseteq \mathsf{An}(\Delta)$.

An argument (Π, ϕ) constitutes a plausible, or tentative, inference ϕ together with the support Π for that inference.

Example 2.5 Let $\Delta = \{\alpha, \neg\alpha, \neg\alpha \rightarrow \beta\}$. Arguments from Δ include $(\{\alpha\}, \alpha)$, $(\{\neg\alpha, \neg\alpha \rightarrow \beta\}, \beta \vee \delta)$, and $(\{\alpha, \gamma\}, \alpha \wedge \gamma\})$. □

Definition 2.6 (Defeat) *Let* (Π, ϕ) *and* (Θ, ψ) *be any arguments constructed from* Δ*. If* $\vdash \phi \leftrightarrow \neg\psi$*, then* (Π, ϕ) *is a rebutting defeater of* (Θ, ψ)*. If* $\gamma \in \Theta$ *and* $\vdash \phi \leftrightarrow \neg\gamma$*, then* (Π, ϕ) *is an undercutting defeater of* (Θ, ψ).

Defeaters for an argument affect the arguments plausibility. A rebutting defeater is a counter-argument directly against a plausible inference, whereas an undercutting defeater is a counter-argument against some of the assumptions used to derive a plausible inference.

Example 2.7 Let $\Delta = \{\beta, \beta \rightarrow \alpha, \gamma, \gamma \rightarrow \neg\alpha, \neg\delta, \neg\delta \vee \neg\gamma\}$. For the argument $(\{\gamma, \gamma \rightarrow \neg\alpha\}, \neg\alpha)$, the argument $(\{\beta, \beta \rightarrow \alpha\}, \alpha)$ is a rebutting defeater, and the argument $(\{\neg\delta, \neg\delta \vee \neg\gamma), \neg\gamma)$ is an undercutting defeater. □

Rebutting defeat, as defined here, is a symmetrical relation. One way of changing this is by use of priorities, such as in specificity (Poole 1985) or as in epistemic entrenchment (Gärdenfors 1988). However, there are a number of issues that deserve further attention before we refine handling of inconsistencies with priorities. (The introduction and use of priorities is discussed in the final section.)

Definition 2.8 *Let* Δ *be a database and* $\Gamma \subseteq \mathsf{An}(\Delta)$. *Then:*

$$\begin{aligned}
&\mathsf{CON}(\Delta) = \{\Pi \subseteq \Delta | \Pi \nvdash \bot\}\\
&\mathsf{INC}(\Delta) = \{\Pi \subseteq \Delta | \Pi \vdash \bot\}\\
&\mathsf{MC}(\Delta) = \{\Pi \in \mathsf{CON}(\Delta) | \forall \Phi \in \mathsf{CON}(\Delta) \cdot \Pi \not\subset \Phi\}\\
&\mathsf{MI}(\Delta) = \{\Pi \in \mathsf{INC}(\Delta) | \forall \Phi \in \mathsf{INC}(\Delta) \cdot \Phi \not\subset \Pi\}\\
&\mathsf{FREE}(\Delta) = \bigcap \mathsf{MC}(\Delta)\\
&\mathsf{MIN}(\Gamma) = \{(\Pi, \phi) \in \Gamma | \forall (\Phi, \phi) \in \Gamma \cdot \Phi \not\subset \Pi\}
\end{aligned}$$

Hence $\mathsf{MC}(\Delta)$ is the set of maximally consistent subsets of Δ; $\mathsf{MI}(\Delta)$ is the set of minimally inconsistent subsets of Δ; $\mathsf{FREE}(\Delta)$ is the set of information that all maximally consistent subsets of Δ have in common; and $\mathsf{MIN}(\Gamma)$ is the set of minimal arguments for a set of arguments.

Lemma 2.9 *Let* Δ *be a database. Then:*

$$\mathsf{MI}(\Delta) = \{\Pi \subseteq \Delta | (\forall \phi \in \Pi \cdot \Pi - \phi \in \mathsf{CON}(\Delta)) \wedge \Pi \notin \mathsf{CON}(\Delta)\}$$

Lemma 2.10 *Let* Π *and* Θ *be databases. Then*

(i) $\mathsf{Cn}(\Pi) \cup \mathsf{Cn}(\Theta) \subseteq \mathsf{Cn}(\Pi \cup \Theta)$.

(ii) $\mathsf{Cn}(\Pi \cap \Theta) \subseteq \mathsf{Cn}(\Pi) \cap \mathsf{Cn}(\Theta)$.

Lemma 2.11

$$\forall \psi \in \mathcal{L} \cdot (\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \psi) \rightarrow (\forall \Theta \in \mathsf{MC}(\Delta) \cdot \Theta \nvdash \neg\psi)$$

Lemma 2.12

$$\forall \Pi \in \mathsf{MC}(\Delta), \Phi \subseteq \Delta \cdot (\Phi \not\subseteq \Pi) \rightarrow (\exists \phi \in \Phi \cdot \Pi \vdash \neg\phi)$$

Proof of 2.12 Let $\Pi \in \mathsf{MC}(\Delta)$, $\Phi \subseteq \Delta$ and $\Phi \not\subseteq \Pi$. Since $\Phi \not\subseteq \Pi$, then $\Phi \neq \emptyset$. Pick $\phi_0 \in \Phi - \Pi$. Assume that $\Pi \nvdash \neg\phi_0$. Then $\Pi \cup \phi_0$ would be consistent, contradicting the maximality of Π. Therefore $\Pi \vdash \neg\phi_0$. □

Lemma 2.13

$$\forall \Pi \in \mathsf{MI}(\Delta), \phi \in \Pi \cdot \Pi - \{\phi\} \vdash \neg\phi$$

Proof of 2.13 Let $\Pi \in \mathsf{MI}(\Delta)$ and $\phi \in \Pi$. Then $(\Pi - \{\phi\}) \cup \{\phi\}$ is inconsistent, and therefore $(\Pi - \{\phi\}) \cup \{\phi\} \vdash \neg\phi$. So, $\Pi - \{\phi\} \vdash \phi \rightarrow \neg\phi$, and therefore also $\Pi - \{\phi\} \vdash \neg\phi$, because $\vdash (\phi \rightarrow \neg\phi) \leftrightarrow \neg\phi$. □

Lemma 2.14

$$\forall \Pi, \Phi \in \mathsf{CON}(\Delta) \cdot (\Pi \cup \Phi \notin \mathsf{CON}(\Delta)) \rightarrow (\exists \phi \in \mathcal{L} \cdot ((\Pi \vdash \neg\phi) \wedge (\Phi \vdash \phi)))$$

Proof of 2.14 Let $\Theta \subseteq \Phi$ be a minimal subset such that $\Pi \cup \Theta \notin \mathsf{CON}(\Delta)$. By compactness of classical logic, Θ is a finite set (since any derivation of $\perp$ only require a finite number of inferences from a finite set of sentences). Let ϕ be the conjunction of all formulas in Θ. Then $\Pi, \phi \vdash \perp$ implying $\Pi \vdash \neg\phi$. We also have $\Phi \vdash \phi$. □

Lemma 2.15

$$\bigcap \mathsf{MC}(\Delta) = \Delta - \bigcup \mathsf{MI}(\Delta)$$

Proof of 2.15 For all $\alpha \in \Delta$, $\alpha \notin \bigcup \mathsf{MI}(\Delta)$ iff $\forall \Phi \in \mathsf{MI}(\Delta) \cdot \alpha \notin \Phi$ iff $\forall \Pi \in \mathsf{CON}(\Delta) \cdot \Pi \nvdash \neg\alpha$ iff $\forall \Pi \in \mathsf{MC}(\Delta) \cdot \Pi \nvdash \neg\alpha$ iff $\forall \Pi \in \mathsf{MC}(\Delta) \cdot \alpha \in \Pi$ iff $\alpha \in \bigcap \mathsf{MC}(\Delta)$. □

We can consider a maximally consistent subset of a database as capturing a "plausible" or "coherent" view on the database. For this reason, the set $\mathsf{MC}(\Delta)$ is important in many of the definitions presented in the next section. Furthermore, we consider $\mathsf{FREE}(\Delta)$, which is equal to $\bigcap \mathsf{MC}(\Delta)$, as capturing all the "uncontroversial" information in Δ. In contrast, we consider the set $\bigcup \mathsf{MI}(\Delta)$ as capturing all the "problematical" data Δ.

Example 2.16 Let $\Delta = \{\alpha, \neg\alpha, \alpha \rightarrow \beta, \neg\alpha \rightarrow \beta, \gamma\}$. This gives two maximally consistent subsets, $\Phi_1 = \{\alpha, \alpha \rightarrow \beta, \neg\alpha \rightarrow \beta, \gamma\}$, and $\Phi_2 = \{\neg\alpha, \alpha \rightarrow \beta, \neg\alpha \rightarrow \beta, \gamma\}$. From this $\bigcap \mathsf{MC}(\Delta)$=$\{\alpha \rightarrow \beta, \neg\alpha \rightarrow \beta, \gamma\}$, and a minimally inconsistent subset $\Psi = \{\alpha, \neg\alpha\}$. □

Lemma 2.17
Let max *be an operator picking* $\subseteq$*-maximal elements from a set of sets.*

$$\mathsf{MC}(\Delta \cup \{\alpha\}) = \{\Phi \in \mathsf{MC}(\Delta) | \Phi \vdash \neg\alpha\} \cup \{\Phi \cup \{\alpha\} | \Phi \in \mathsf{max}\{\Psi \in \mathsf{CON}(\Delta) | \Psi \nvdash \neg\alpha\}\}$$

Proof of 2.17 Let A1 denote "$\Pi \in \mathsf{MC}(\Delta \cup \{\alpha\})$", A2 denote "$\Pi \in \{\Phi \in \mathsf{MC}(\Delta) | \Phi \vdash \neg\alpha\}$", and A3 denote "$\Pi \in \{\Phi \cup \{\alpha\} | \Phi \in \mathsf{max}\{\Psi \in \mathsf{CON}(\Delta) | \Psi \not\vdash \neg\alpha\}\}$". The proof has two main cases. To prove A1 implies (A2 or A3), let A1 be the case and assume that neither of A2 and A3 is the case. If $\Pi \vdash \neg\alpha$, then an easy argument shows that A2 holds. Therefore we assume $\Pi \not\vdash \neg\alpha$, and thus $\alpha \in \Pi$, because Π is maximal consistent. If $\Pi \subseteq \Delta$ (i.e. $\alpha \in \Delta$), then Π is maximal in $\mathsf{CON}(\Delta)$, which is in conflict with the assumptions. Therefore, $\alpha \notin \Delta$, but then $\Pi - \{\alpha\} \subseteq \Delta$, and $\Pi - \{\alpha\}$ is maximal in $\{\Psi \in \mathsf{CON}(\Delta) | \Psi \not\vdash \neg\alpha\}$, because otherwise Π could not be in $\mathsf{MC}(\Delta \cup \{\alpha\})$. But this is in conflict with the assumptions, and therefore we can conclude that either A2 or A3. To prove (A2 or A3) implies A1, let (A2 or A3) be the case, and assume that A1 is not the case. If A2 is the case, then $\Pi \in \mathsf{MC}(\Delta)$ and Π is not consistent with α. Thus, A1 is the case, contradicting the assumption. If A3 is the case, then for some Φ, it is the case that $\Phi \cup \{\alpha\} = \Pi$ and that this Φ is maximal in $\{\Psi \in \mathsf{CON}(\Delta) | \Psi \not\vdash \neg\alpha\}$. Since Π is not maximal in $\mathsf{CON}(\Delta \cup \{\alpha\})$, then for some $\beta \in \Delta - \Pi$, it is the case that $\Phi \cup \{\alpha, \beta\} \not\vdash \neg\alpha$. But, this contradicts the maximality of Φ, because $\beta \notin \Phi$ and $\Phi \cup \{\beta\} \not\vdash \neg\alpha$. Therefore, we can conclude that A1 is the case. □

Example 2.18 [Application of Lemma 2.17] Let $\Delta = \{\alpha, \gamma \wedge (\alpha \vee \neg\beta), \neg\gamma \wedge (\neg\alpha \vee \neg\beta)\}$. Then $\mathsf{MC}(\Delta) = \{\{\alpha, \gamma \wedge (\alpha \vee \neg\beta)\}, \{\alpha, \neg\gamma \wedge (\neg\alpha \vee \neg\beta)\}\}$. And $\mathsf{MC}(\Delta \cup \{\beta\}) = \{\{\alpha, \gamma \wedge (\alpha \vee \neg\beta), \beta\}, \{\alpha, \neg\gamma \wedge (\neg\alpha \vee \neg\beta)\}, \{\neg\gamma \wedge (\neg\alpha \vee \neg\beta), \beta\}\}$. This example shows that $\mathsf{MC}(\Delta \cup \{\beta\})$ cannot be constructed directly from $\mathsf{MC}(\Delta)$. □

As immediate consequences of Lemma 2.17 we get the following properties.

Lemma 2.19

(i) $\forall \alpha \in \mathcal{L}, \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \cup \{\alpha\} \in \mathsf{MC}(\Delta \cup \{\alpha\}) \leftrightarrow \Phi \not\vdash \neg\alpha$

(ii) $\forall \alpha \in \mathcal{L}, \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \neg\alpha \rightarrow \Phi \in \mathsf{MC}(\Delta \cup \{\alpha\})$

(iii) $\forall \alpha \in \mathcal{L}$ s.t. $\alpha \not\vdash \bot \cdot (\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \neg\alpha) \rightarrow (\mathsf{MC}(\Delta \cup \{\alpha\}) = \mathsf{MC}(\Delta) \cup \{\{\alpha\}\})$

(iv) $\forall \alpha \in \mathcal{L}, \Phi \in \mathsf{MC}(\Delta) \cdot \exists \Psi \in \mathsf{MC}(\Delta \cup \{\alpha\}) \cdot \Phi \subseteq \Psi$

Lemma 2.20

$$\mathsf{FREE}(\Delta \cup \{\alpha\}) \subseteq \mathsf{FREE}(\Delta) \cup \{\alpha\}$$

Proof of 2.20 By applying the definition of FREE and Lemma 2.17, $\mathsf{FREE}(\Delta \cup \{\alpha\})$ is equal to: (*) $\bigcap(\{\Phi \in \mathsf{MC}(\Delta) | \Phi \vdash \neg\alpha\} \cup \{\Phi \cup \{\alpha\} | \Phi \in \mathsf{max}\{\Psi \in \mathsf{CON}(\Delta) | \Psi \not\vdash \neg\alpha\}\})$. Now, there are two cases to consider. (1) If $\Phi \vdash \neg\alpha$ for no $\Phi \in \mathsf{MC}(\Delta)$, then (*) is equal to (using the obvious property: $\mathsf{MC}(\Delta) = \mathsf{max}(\mathsf{CON}(\Delta))$):

$$\begin{aligned} & \bigcap\{\Phi \cup \{\alpha\} | \Phi \in \mathsf{max}(\mathsf{CON}(\Delta))\} \\ = & \bigcap\{\Phi \cup \{\alpha\} | \Phi \in \mathsf{MC}(\Delta)\} \\ = & \{\alpha\} \cup \bigcap \mathsf{MC}(\Delta) \\ = & \mathsf{FREE}(\Delta) \cup \{\alpha\}. \end{aligned}$$

(2) If $\Phi \vdash \neg\alpha$ for some $\Phi \in \mathsf{MC}(\Delta)$, then (*) is equal to $\bigcap(A \cup C)$, where

$$A = \{\Phi \mid \Phi \in \mathsf{MC}(\Delta) \wedge \Phi \vdash \neg\alpha\}$$

$$B = \{\Phi \mid \Phi \in \mathsf{MC}(\Delta) \wedge \Phi \not\vdash \neg\alpha\}$$

$$C = \{\Phi \cup \{\alpha\} \mid \Phi \in \mathsf{max}\{\Psi \in \mathsf{CON}(\Delta) \mid \Psi \not\vdash \alpha\}\}$$

$$D = \{\Phi \mid \Phi \in \mathsf{max}\{\Psi \in \mathsf{CON}(\Delta) \mid \Psi \not\vdash \alpha\}\}$$

Since for all $\Phi \in A$, $\Phi \vdash \neg\alpha$ holds, α can not be in the intersection of $A \cup C$. Therefore, $\bigcap(A \cup C) = \bigcap(A \cup D)$. We also have $\cap(\mathsf{MC}(\Delta)) = \cap(A \cup B)$. Furthermore, it is straightforward to show $B \subseteq D$ holds. By using a standard property of set theory, we can therefore show $\bigcap(A \cup D) \subseteq \bigcap(A \cup B)$. Hence we have $\bigcap(\mathsf{MC}(\Delta \cup \{\alpha\})) \subseteq \bigcap(\mathsf{MC}(\Delta))$. □

As we discuss in section 4, these results have ramifications in deriving inferences from $\mathsf{FREE}(\Delta)$, since the choice of updating (in the form of either $\mathsf{FREE}(\Delta \cup \{\alpha\})$ or $\mathsf{FREE}(\Delta) \cup \{\alpha\}$) can affect the reasoning.

3 The argumentative structure A^*

For a database Δ, we define an argumentative structure to be any set of subsets of $\mathsf{An}(\Delta)$. The intention behind the definition for an argumentative structure is that different subsets of $\mathsf{An}(\Delta)$ have different degrees of acceptability. Below, we define one particular argumentative structure A^*, and then explain how the definition captures notions of acceptability.

Evidently from the definition of an argumentative structure, a whole range of different structures can be defined. However, we see A^*as being distinguished, because it is defined from some very basic concepts of classical logic. These are the concepts of consistent subsets, maximal consistent subsets and free subsets as defined in the previous section.

Definition 3.1 (The argumentative structure A^*) *Let Δ be a database. Then,*

$$\begin{array}{lll}
\mathsf{AT}(\Delta) & = & \{(\emptyset,\phi) | \emptyset \vdash \phi\} \\
\mathsf{AF}(\Delta) & = & \{(\Pi,\phi) | \Pi \subseteq \mathsf{FREE}(\Delta) \wedge \Pi \vdash \phi\} \\
\mathsf{AB}(\Delta) & = & \{(\Pi,\phi) | \Pi \in \mathsf{CON}(\Delta) \wedge \Pi \vdash \phi \wedge (\forall \Phi \in \mathsf{MC}(\Delta), \psi \in \Pi \cdot \Phi \vdash \psi)\} \\
\mathsf{ARU}(\Delta) & = & \{(\Pi,\phi) | \Pi \in \mathsf{CON}(\Delta) \wedge \Pi \vdash \phi \wedge \\
 & & \quad (\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \not\vdash \neg\phi) \wedge (\forall \Phi \in \mathsf{MC}(\Delta), \psi \in \Pi \cdot \Phi \not\vdash \neg\psi)\} \\
\mathsf{AU}(\Delta) & = & \{(\Pi,\phi) | \Pi \in \mathsf{CON}(\Delta) \wedge \Pi \vdash \phi \wedge (\forall \Phi \in \mathsf{MC}(\Delta), \psi \in \Pi \cdot \Phi \not\vdash \neg\psi)\} \\
\mathsf{A}\forall(\Delta) & = & \{(\Pi,\phi) | \Pi \in \mathsf{CON}(\Delta) \wedge \Pi \vdash \phi \wedge (\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \phi)\} \\
\mathsf{AR}(\Delta) & = & \{(\Pi,\phi) | \Pi \in \mathsf{CON}(\Delta) \wedge \Pi \vdash \phi \wedge (\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \not\vdash \neg\phi)\} \\
\mathsf{A}\exists(\Delta) & = & \{(\Pi,\phi) | \Pi \in \mathsf{CON}(\Delta) \wedge \Pi \vdash \phi\}
\end{array}$$

The naming conventions for the argument sets are motivated as follows. T is for the tautological arguments - i.e. those that follow from the empty set of premises. F is for the free arguments - (due to Benferhat et al (1993)) - which are the arguments that follow from the data that is free of inconsistencies. B is for the backed arguments - i.e. those for which all the premises follow from all the maximally consistent subsets of the data. RU is for the arguments that are not subject to either rebutting or undercutting. U is for the arguments that are not subject to undercutting. $\forall$ is for the universal arguments - (essentially due to Manor and Rescher (1970)), where it was called inevitable arguments) - which are the arguments that follow from all maximally consistent subsets of the data. R is for the arguments that are not subject to rebutting. $\exists$ is for existential arguments - (essentially due to Manor and Rescher (1970)) - which are the arguments with consistent premises.

The definitions for $\mathsf{A}\exists$, AF, AT should be clear. We therefore focus on the remainder. AR allows an argument (Π, ϕ) only if there is no maximally consistent subset that gives $\neg\phi$. AU allows an argument (Π, ϕ) only if for all items ψ in Π, there is no maximally consistent subset that gives $\neg\psi$. ARU combines the conditions of the AR and AU. Notice that AR and $\mathsf{A}\forall$ have very similar definitions, with the only difference being "$\Phi \not\vdash \neg\phi$" in AR versus "$\Phi \vdash \phi$" in $\mathsf{A}\forall$. A similar remark applies to AU and AB. Therefore $\mathsf{A}\forall$ and AB are strengthenings of AR and AU, respectively (i.e. "$\not\vdash \neg\phi$" replaced with "$\vdash \phi$").

Example 3.2 We give an example of a database, and some of the items in each argument set. Take $\Delta = \{\alpha, \neg\alpha\}$. Then $(\{\alpha, \neg\alpha\}, \alpha \wedge \neg\alpha) \in \mathsf{An}(\Delta)$, $(\{\alpha\}, \alpha) \in \mathsf{A}\exists(\Delta)$, $(\{\alpha\}, \alpha \vee \beta) \in \mathsf{AR}(\Delta)$, if $\beta \not\vdash \alpha$, $(\{\}, \alpha \vee \neg\alpha) \in \mathsf{A}\forall(\Delta)$. Furthermore, $\mathsf{A}\forall(\Delta) = \mathsf{AF}(\Delta) = \mathsf{AB}(\Delta) = \mathsf{ARU}(\Delta) = \mathsf{AU}(\Delta) = \mathsf{AT}(\Delta)$. □

Example 3.3 As another example, consider $\Delta = \{\neg\alpha \wedge \beta, \alpha \wedge \beta\}$. Then for $\Pi = \{\alpha \wedge \beta\}$, $(\Pi, \beta) \in \mathsf{A}\exists(\Delta)$, $(\Pi, \beta) \in \mathsf{AR}(\Delta)$, and $(\Pi, \beta) \in \mathsf{A}\forall(\Delta)$. But there is no $\Pi \subseteq \Delta$ such that $(\Pi, \beta) \in \mathsf{AU}(\Delta)$, $(\Pi, \beta) \in \mathsf{ARU}(\Delta)$, $(\Pi, \beta) \in \mathsf{AB}(\Delta)$, or $(\Pi, \beta) \in \mathsf{AF}(\Delta)$. □

Proposition 3.4

$\mathsf{AT}(\Delta) \subseteq \mathsf{AF}(\Delta) = \mathsf{AB}(\Delta) = \mathsf{ARU}(\Delta) = \mathsf{AU}(\Delta) \subseteq \mathsf{A\forall}(\Delta) \subseteq \mathsf{AR}(\Delta) \subseteq \mathsf{A\exists}(\Delta) \subseteq \mathsf{An}(\Delta)$

Proof of 3.4 We give a proof for a sufficient number of cases to establish the proposition. We use the definitions of the various sets of arguments without explicit reference.

$\mathsf{AT}(\Delta) \subseteq \mathsf{AF}(\Delta)$: Since $\emptyset \subseteq \mathsf{FREE}(\Delta)$ for any Δ, the set of tautological arguments is definable as $\{(\Pi, \phi) \in \mathsf{AF}(\Delta) | \Pi = \emptyset\}$.

$\mathsf{AF}(\Delta) \not\subseteq \mathsf{AT}(\Delta)$: To construct a counterexample, let the database be $\Delta = \{\alpha\}$, containing a single contingent sentence. Then $(\Delta, \alpha) \in \mathsf{AF}(\Delta)$, but $(\Delta, \alpha) \notin \mathsf{AT}(\Delta)$.

$\mathsf{AB}(\Delta) \subseteq \mathsf{AF}(\Delta)$: Assume (i) that $(\Pi, \phi) \notin \mathsf{AF}(\Delta)$ to prove (ii) that $(\Pi, \phi) \notin \mathsf{AB}(\Delta)$. If $\Pi \not\vdash \phi$, then (ii) follows. Otherwise $\Pi \vdash \phi$ and $\Pi \not\subseteq \mathsf{FREE}(\Delta)$. Pick a $\Phi_0 \in \mathsf{MC}(\Delta)$ such that $\Pi \not\subseteq \Phi_0$. By application of Lemma 2.12 it follows that $\exists \psi \in \Pi \cdot \Phi_0 \vdash \neg\psi$. From this (ii) follows, because we now have $\neg\forall\Phi \in \mathsf{MC}(\Delta), \psi \in \Pi \cdot \Phi \vdash \psi$.

$\mathsf{AF}(\Delta) \subseteq \mathsf{AB}(\Delta)$: Assume (ii) to prove (i). Either $\Pi \not\vdash \phi$ or $\Pi \notin \mathsf{CON}(\Delta)$, in which cases (i) follow, or $\Pi \vdash \phi$ and $\Pi \in \mathsf{CON}(\Delta)$ and we can pick a Φ_0 and ψ_0 such that $\Phi_0 \in \mathsf{MC}(\Delta)$, $\psi_0 \in \Pi$ and $\Phi_0 \not\vdash \psi_0$. By assumption $\psi_0 \notin \mathsf{Cn}(\Phi_0)$ and from the standard $\cap$-properties we get $\psi_0 \notin \bigcap_{\Phi \in \mathsf{MC}(\Delta)} \mathsf{Cn}(\Phi)$. Using Lemma 2.10 gives $\psi_0 \notin \mathsf{Cn}(\bigcap_{\Phi \in \mathsf{MC}(\Delta)} \Phi)$. From here (i) follows, since $\Pi \not\subseteq \mathsf{FREE}(\Delta)$.

$\mathsf{AU}(\Delta) \subseteq \mathsf{AB}(\Delta)$: Assume (ii) to prove (iii) $(\Pi, \phi) \notin \mathsf{AU}(\Delta)$. Either $\Pi \notin \mathsf{CON}(\Delta)$ or $\Pi \not\vdash \phi$, in which cases (iii) follow, or $\Pi \in \mathsf{CON}(\Delta)$ and $\Pi \vdash \phi$ and we can pick some Φ_0 and ψ_0 such that $\Phi_0 \in \mathsf{MC}(\Delta)$, $\psi_0 \in \Pi$ and $\Phi_0 \not\vdash \psi_0$. Since $\Pi \vdash \psi_0$ and $\Phi_0 \not\vdash \psi_0$, $\Pi \cup \Phi_0 \notin \mathsf{CON}(\Delta)$, because otherwise Φ_0 would not have been maximal consistent. By Lemma 2.12, since $\Pi \not\subseteq \Phi_0$, there is some $\psi_1 \in \Pi$, such that $\Phi_0 \vdash \neg\psi_1$. From this (iii) follows.

$\mathsf{AB}(\Delta) \subseteq \mathsf{AU}(\Delta)$: This follows from Lemma 2.11.

$\mathsf{ARU}(\Delta) = \mathsf{AU}(\Delta)$: $\mathsf{ARU}(\Delta)$ is definable as $\mathsf{AR}(\Delta) \cap \mathsf{AU}(\Delta)$, and since $\mathsf{AU}(\Delta) \subseteq \mathsf{AR}(\Delta)$, cf. below, this is equivalent to $\mathsf{AU}(\Delta)$.

$\mathsf{A\forall}(\Delta) \not\subseteq \mathsf{AU}(\Delta)$: Let $\Delta = \{\alpha \wedge \beta, \neg\alpha \wedge \beta\}$. Then $(\Delta, \beta) \in \mathsf{A\forall}(\Delta)$, but $(\Delta, \beta) \notin \mathsf{AU}(\Delta)$.

$\mathsf{AU}(\Delta) \subseteq \mathsf{A\forall}(\Delta)$: Assume (iv) $(\Pi, \phi) \notin \mathsf{A\forall}(\Delta)$ to prove (iii). Either $\Pi \notin \mathsf{MC}(\Delta)$ or $\Pi \not\vdash \phi$, in which cases (iv) follow, or $\Pi \in \mathsf{MC}(\Delta)$ and $\Pi \vdash \phi$ and we can pick some Φ_0 such that $\Phi_0 \in \mathsf{MC}(\Delta)$ and $\Phi_0 \not\vdash \phi$. Assume, to prove a

$$\mathsf{AT}(\Delta)$$
$$|$$
$$\mathsf{AF}(\Delta) = \mathsf{AB}(\Delta) = \mathsf{ARU}(\Delta) = \mathsf{AU}(\Delta)$$
$$|$$
$$\mathsf{A}\forall(\Delta)$$
$$|$$
$$\mathsf{AR}(\Delta)$$
$$|$$
$$\mathsf{A}\exists(\Delta)$$
$$|$$
$$\mathsf{An}(\Delta)$$

Figure 1: Partial order on A^* induced by $\subseteq$

contradiction, that (v) $\forall\Theta \in \mathsf{MC}(\Delta), \psi \in \Pi \cdot \Theta \not\vdash \neg\psi$. Using Lemma 2.12, as in a previous subcase, we can find $\psi_0 \in \Pi$ such that $\Phi_0 \vdash \neg\psi_0$. This contradicts (v) and we thus we have proved its negation. From the negation of (v) it immediately follows that (iii) holds.

$\mathsf{A}\forall(\Delta) \subseteq \mathsf{AR}(\Delta)$: This follows from Lemma 2.11.

$\mathsf{AR}(\Delta) \not\subseteq \mathsf{A}\forall(\Delta)$: Let $\Delta = \{\alpha, \neg\alpha \wedge \beta\}$. Then $(\Delta, \beta) \in \mathsf{AR}(\Delta)$, but $(\Delta, \beta) \notin \mathsf{A}\forall(\Delta)$.

$\mathsf{AR}(\Delta) \subseteq \mathsf{A}\exists(\Delta)$: $\mathsf{AR}(\Delta)$ is definable as $\{(\Pi, \phi) \in \mathsf{A}\exists(\Delta) | \forall\Phi \in \mathsf{MC}(\Delta) \cdot \Phi \not\vdash \neg\phi\}$.

$\mathsf{A}\exists(\Delta) \not\subseteq \mathsf{AR}(\Delta)$: Let $\Delta = \{\alpha, \neg\alpha\}$. Then $(\Delta, \alpha) \in \mathsf{A}\exists(\Delta)$, but $(\Delta, \alpha) \notin \mathsf{AR}(\Delta)$.

$\mathsf{A}\exists(\Delta) \subseteq \mathsf{An}(\Delta)$: $\mathsf{A}\exists(\Delta)$ is definable as $\{(\Pi, \phi) \in \mathsf{An}(\Delta) | \Pi \in \mathsf{CON}(\Delta)\}$.

$\mathsf{An}(\Delta) \not\subseteq \mathsf{A}\exists(\Delta)$: Let $\Delta = \{\alpha \wedge \neg\alpha\}$. Then $(\Delta, \beta) \in \mathsf{An}(\Delta)$, but $(\Delta, \beta) \notin \mathsf{A}\exists(\Delta)$.

□

We summarize this result by the diagram in Figure 1. The main features to notice are that A^* is a linear structure, and that there is an equivalence of AF, AB, ARU, and AU. However note that the definition of A^* is based on the classical consequence relation and on concepts related to classical logic. If we changed the underlying logic to, say, intuitionistic logic or relevance logic, then we would have a completely different basis for the hierarchy. Another possibility would be to extend the underlying classical logic to allow the use of defaults. And yet another is to add extra constraints in the form of priorities over the items in the database. Whereas the two former suggestions mainly relate to the notion of argument, the latter relates to the notion of acceptability that would have to be defined for the suggested argumentative structures.

We see the argumentative structure A^* as distinguished, because it is based on a notion of argument and defeat that is frequently used in the literature, and its definition is based on intuitive concepts rising naturally from the context of classical logic.

4 Argumentative logics induced by A^*

Each argument set in A^* induces a consequence relation. In the following we let "x" syntactically denote an arbitrary member of the suffixes: "T, F, B, RU, U, $\forall$, R, $\exists$, n".

Definition 4.1 *An* x*-consequence closure is denoted* Cx*, and defined as follows,*

$$\mathsf{Cx}(\Delta) = \{\phi | \exists \Pi \subseteq \Delta \cdot (\Pi, \phi) \in \mathsf{Ax}(\Delta)\}$$

Definition 4.2 *An* x*-consequence relation is denoted* $\vdash_x$*, and defined as follows,*

$$\Delta \vdash_x \phi \textit{ iff } \phi \in \mathsf{Cx}(\Delta)$$

$\vdash_n$ is classical entailment, but we continue to omit the subscript.

Proposition 4.3

$$\mathsf{CT}(\Delta) \subseteq \mathsf{CF}(\Delta) = \mathsf{CB}(\Delta) = \mathsf{CRU}(\Delta) = \mathsf{CU}(\Delta) \subseteq \mathsf{C}\forall(\Delta) \subseteq \mathsf{CR}(\Delta) \subseteq \mathsf{C}\exists(\Delta) \subseteq \mathsf{Cn}(\Delta)$$

Proof of 4.3 Follows immediately from Definition 4.1 and Proposition 3.4. □

Proposition 4.4 *If* Δ *is consistent, then for all* $\mathsf{x} \neq \mathsf{T}$, $\mathsf{Cx}(\Delta) = \mathsf{Cn}(\Delta)$*.*

Proof of 4.4 If Δ is consistent, then $\mathsf{FREE}(\Delta) = \Delta$ and therefore $\mathsf{CF}(\Delta) = \mathsf{Cn}(\Delta)$. From this and Proposition 4.3 the result follows. □

Definition 4.5 *If* $\phi \in \mathsf{Cx}(\Delta)$*, then* Δ x*-derives* ϕ*. If either* $\phi \in \mathsf{Cx}(\Delta)$ *or* $\neg\phi \in \mathsf{Cx}(\Delta)$ *then* Δ x*-decides* ϕ*. If* Δ *does not* x*-decide* ϕ*, then* ϕ *is* x*-undecided by* Δ*.*

A minimality requirement can be added to arguments, without changing the properties of the consequence relations.

Proposition 4.6 $\{\phi | \exists \Pi \cdot (\Pi, \phi) \in \mathsf{Ax}(\Delta)\} = \{\phi | \exists \Pi \cdot (\Pi, \phi) \in \mathsf{MIN}(\mathsf{Ax}(\Delta))\}$

Proof of 4.6 The $\subseteq$-ordering is well-founded and therefore minimal elements can always be found. □

Lemma 4.7 *$\alpha \in \mathsf{Cx}(\Delta)$ and $\alpha \vdash \beta$ imply $\beta \in \mathsf{Cx}(\Delta)$.*

Proof of 4.7 The proof is by cases for each consequence relation. Assume in general that (i) $\alpha \in \mathsf{Cx}(\Delta)$, (ii) $\alpha \vdash \beta$ and (iii) $\beta \notin \mathsf{Cx}(\Delta)$, to derive a contradiction.

CF: (i) ensures the existence of some $\Pi_0 \subseteq \mathsf{FREE}(\Delta)$ such that $\Pi_0 \vdash \alpha$. Using (ii) this also gives $\Pi_0 \vdash \beta$, contradicting (iii).

C$\forall$: This case is similar to the previous case, using $\Pi_0 \in \mathsf{MC}(\Delta)$ instead.

CR: Briefly, since (i), (ii) and (iii) there must be some $\Phi_0 \in \mathsf{CON}(\Delta)$ such that $\Phi_0 \vdash \neg\beta$. Using contraposition and (ii) this gives $\Phi_0 \vdash \neg\alpha$. This is impossible.

The cases for CT, C$\exists$ and Cn are simple and we have omitted them. The cases for CB, CRU and CU are covered by the case for CF because of the result established as Proposition 4.3. □

4.1 Standard properties of the consequence relations

The following standard properties of consequence relations have been adapted from those given by Gabbay 1985 and Gärdenfors and Makinson (1993).

Definition 4.8 *Let $\vdash_x$ be some consequence relation. We introduce the following properties:*

$\Delta \vdash_x \alpha$ *if* $\Delta \vdash \alpha$	*(Supraclassicality)*
$\Delta \cup \{\alpha\} \vdash_x \alpha$	*(Reflexivity)*
$\Delta \cup \{\beta\} \vdash_x \gamma$ *if* $\Delta \cup \{\alpha\} \vdash_x \gamma$ *and* $\vdash \alpha \leftrightarrow \beta$	*(Left logical equivalence)*
$\Delta \vdash_x \alpha$ *if* $\Delta \vdash_x \beta$ *and* $\vdash \beta \rightarrow \alpha$	*(Right weakening)*
$\Delta \vdash_x \alpha \wedge \beta$ *if* $\Delta \vdash_x \alpha$ *and* $\Delta \vdash_x \beta$	*(And)*
$\Delta \cup \{\alpha\} \vdash_x \beta$ *if* $\Delta \not\vdash_x \neg\alpha$ *and* $\Delta \vdash_x \beta$	*(Rational monotonicity)*
$\Delta \cup \{\alpha\} \vdash_x \beta$ *if* $\Delta \vdash_x \alpha$ *and* $\Delta \vdash_x \beta$	*(Cautious monotonicity)*
$\Delta \cup \{\alpha\} \vdash_x \beta$ *if* $\Delta \vdash_x \beta$	*(Monotonicity)*
$\Delta \vdash_x \beta$ *if* $\Delta \vdash_x \alpha$ *and* $\Delta \cup \{\alpha\} \vdash_x \beta$	*(Cut)*
$\Delta \vdash \bot$ *if* $\Delta \vdash_x \bot$	*(Consistency preservation)*
$\Delta \vdash_x \alpha \rightarrow \beta$ *if* $\Delta \cup \{\alpha\} \vdash_x \beta$	*(Conditionalization)*
$\Delta \cup \{\alpha\} \vdash_x \beta$ *if* $\Delta \vdash_x \alpha \rightarrow \beta$	*(Deduction)*
$\Delta \cup \{\alpha \vee \beta\} \vdash_x \gamma$ *if* $\Delta \cup \{\alpha\} \vdash_x \gamma$ *and* $\Delta \cup \{\beta\} \vdash_x \gamma$	*(Or)*

These properties have been proposed as desirable conditions of a consequence relation. In particular, identifying the properties that fail indicates the deviation from well-understood and intuitive formalisms such as classical logic.

Proposition 4.9 *Supraclassicality fails for all the* x*-consequence relations, except for the* n*-consequence relation.*

Proof of 4.9 Let $\Delta = \{\alpha \wedge \neg\alpha\}$, then $\Delta \vdash \beta$, but $\Delta \not\vdash_x \beta$. For n, obvious. □

Proposition 4.10 *Reflexivity holds for the* $\exists$*-consequence relation only if* $\alpha \not\vdash \bot$. *Reflexivity fails for the* R, $\forall$, U, RU, F *and* T*-consequence relations. Reflexivity holds for the* n*-consequence relation.*

Proof of 4.10 Consider the database $\{\bot\}$ to see that reflexivity fails for all but n-consequence. □

Proposition 4.11 *Left logical equivalence succeeds for all* x*-consequence relations.*

Proof of 4.11 This is a consequence of the fact that the argumentative logics are insensitive to the logical form of the items in a database. □

Proposition 4.12 *Right weakening succeeds for all* x*-consequence relations.*

Proof of 4.12 See Lemma 4.7 □

Proposition 4.13 *And fails for the* $\exists$ *and the* R*-consequence relations. And succeeds for the* $\forall$, U, RU, F, B, T *and* n*-consequence relations.*

Proof of 4.13 For $\exists$, take $\Delta = \{\alpha, \neg\alpha\}$. $\alpha \wedge \neg\alpha$ is not an $\exists$-consequence. For R, use Δ again. If β is R-undecided, then $\alpha \vee \beta$ and $\neg\alpha \vee \beta$ are R-consequences, but their conjunction is not. For $\forall$, assume $\Delta \vdash_\forall \alpha$, and $\Delta \vdash_\forall \beta$. Therefore, $\forall\Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \alpha$, and $\forall\Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \beta$. Using the classical and-property on each $\Phi \in \mathsf{MC}(\Delta)$, we get $\forall\Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \alpha \wedge \beta$. For F, assume $\Delta \vdash_F \alpha$, and $\Delta \vdash_F \beta$. Therefore, $\mathsf{FREE}(\Delta) \vdash \alpha$, and $\mathsf{FREE}(\Delta) \vdash \beta$, and so, $\mathsf{FREE}(\Delta) \vdash \alpha \wedge \beta$. Hence, $\Delta \vdash_F \alpha \wedge \beta$, and hence also for U, RU and B. For T and n the and property is obvious. □

Proposition 4.14 *Rational monotonicity fails for* R, $\forall$, F, U, RU, B. *Rational monotonicity succeeds for* n, $\exists$, T.

Proof of 4.14 For R, let $\Delta = \{\gamma \wedge \beta, \neg\gamma \wedge (\alpha \rightarrow \neg\beta)\}$. For $\forall$, F, let $\Delta = \{\gamma, \neg\gamma \wedge (\alpha \rightarrow \neg\beta), \beta\}$. The latter counterexample also counts for U, RU, B. For n, $\exists$, T, obvious. □

Proposition 4.15 *Cautious monotonicity succeeds for all* x*-consequence relations, except for the* R*-consequence relation.*

Proof of 4.15 For $\exists$, obvious. For R, let $\Delta = \{\gamma \wedge \beta \wedge \alpha, \neg\gamma \wedge (\alpha \rightarrow \neg\beta)\}$. For $\forall$, assume $\Delta \vdash_{\forall} \alpha$ and $\Delta \vdash_{\forall} \beta$, so $\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \alpha$ and $\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \beta$. Therefore, for any $\Phi \in \mathsf{MC}(\Delta \cup \{\alpha\})$, $\Phi \vdash \alpha$ and $\Phi \vdash \beta$. Therefore, $\Delta \cup \{\alpha\} \vdash_{\forall} \beta$. For F, assume $\Delta \vdash_F \alpha$ and $\Delta \vdash_F \beta$, so $\mathsf{FREE}(\Delta) \vdash \alpha$ and $\mathsf{FREE}(\Delta) \vdash \beta$. Hence, $\mathsf{FREE}(\Delta \cup \{\alpha\}) \vdash \beta$, and so $\Delta \cup \{\alpha\} \vdash_F \beta$, and hence so for $\mathsf{U}, \mathsf{RU}, \mathsf{B}$. For T and n, obvious. $\Box$

Proposition 4.16 *Monotonicity succeeds for the* $\mathsf{n}, \exists, \mathsf{T}$*-consequence relations. Monotonicity fails for the* $\mathsf{R}, \forall, \mathsf{U}, \mathsf{RU}, \mathsf{B}, \mathsf{F}$*-consequence relations.*

Proof of 4.16 For n, $\exists$ and T, obvious. For $\forall$, U, RU, B, and F, take $\Delta = \{\neg\alpha\}$, then $\neg\alpha$ is a consequence of Δ, but not of $\Delta \cup \{\alpha\}$. $\Box$

Proposition 4.17 *Cut fails for the* $\exists$ *and* R*-consequence relations. Cut succeeds for the* n, $\forall$, U, RU, F, B *and* T*-consequence relations.*

Proof of 4.17 For $\exists$ and R, take $\Delta = \{\delta, \neg\delta, \delta \rightarrow \alpha, \neg\delta \rightarrow (\alpha \rightarrow \beta)\}$. For $\forall$, assume $\Delta \vdash_{\forall} \alpha$, and $\Delta \cup \{\alpha\} \vdash_{\forall} \beta$. Since, $\forall \Phi \in \mathsf{MC}(\Delta) \cdot \Phi \vdash \alpha$, then $\mathsf{MC}(\Delta \cup \{\alpha\}) = \{\Phi \cup \{\alpha\} | \Phi \in \mathsf{MC}(\Delta)\}$. Hence, by the assumption we have $\Phi \vdash \alpha$ and $\Phi \cup \{\alpha\} \vdash \beta$ and therefore, using classic cut, $\Phi \vdash \beta$ for any $\Phi \in \mathsf{MC}(\Delta)$. Hence $\Delta \vdash_{\forall} \beta$. For F, assume $\Delta \vdash_F \alpha$, and $\Delta \cup \{\alpha\} \vdash_F \beta$. Therefore $\mathsf{FREE}(\Delta) \vdash \alpha$, and hence $\mathsf{Cn}(\mathsf{FREE}(\Delta)) = \mathsf{Cn}(\mathsf{FREE}(\Delta \cup \{\alpha\}))$. So since $\beta \in \mathsf{Cn}(\mathsf{FREE}(\Delta \cup \{\alpha\}))$, then $\mathsf{FREE}(\Delta) \vdash \beta$ holds, and therefore $\Delta \vdash_F \beta$, and hence so for $\mathsf{U}, \mathsf{RU}, \mathsf{B}$. $\Box$

Proposition 4.18 *Consistency preservation succeeds for all the* x*-consequence relations.*

Proof of 4.18 Consider the contrapositive of the consistency preservation property. For this the consequent holds for all x-consequence relations, and hence the property holds. $\Box$

Proposition 4.19 *Conditionalization succeeds for all the* x*-consequence relations.*

Proof of 4.19 For $\exists$, assume $\Delta \cup \{\alpha\} \vdash_{\exists} \beta$. Then for some $\Pi \in \mathsf{CON}(\Delta \cup \{\alpha\})$, $\Pi \vdash \beta$. Clearly $\Pi - \{\alpha\} \in \mathsf{CON}(\Delta)$ and since $(\Pi - \{\alpha\}) \cup \{\alpha\} \vdash \beta$, then $\Pi - \{\alpha\} \vdash \alpha \rightarrow \beta$ and thus $\Delta \vdash_{\exists} \alpha \rightarrow \beta$.

For R, assume $\Delta \cup \{\alpha\} \vdash_R \beta$. Then for some $\Pi \in \mathsf{CON}(\Delta \cup \{\alpha\})$, $\Pi \vdash \beta$ and for any $\Phi \in \mathsf{MC}(\Delta \cup \{\alpha\})$, $\Phi \not\vdash \neg\beta$. Since $\Phi \not\vdash \neg\beta$ it is, using the property (iv) from Lemma 2.19, also the case for any member in $\mathsf{MC}(\Delta)$, because each of these will be a subset of some member in $\mathsf{MC}(\Delta \cup \{\alpha\})$. Therefore, for all $\Phi \in \mathsf{MC}(\Delta)$, $\Phi \not\vdash \alpha \wedge \neg\beta$. Thus no rebutting argument for the implication $\alpha \rightarrow \beta$ can be constructed, and using an argument similar to the one for the $\exists$-case a supporting argument can be constructed, and we get $\Delta \vdash_R \alpha \rightarrow \beta$.

For $\forall$, assume $\Delta \cup \{\alpha\} \vdash_\forall \beta$. Then (1) for any $\Phi \in \mathsf{MC}(\Delta \cup \{\alpha\})$, $\Phi \vdash \beta$. Let $\Psi \in \mathsf{MC}(\Delta)$. Suppose (2), that $\Psi \cup \{\alpha\} \not\vdash \beta$. Then $\Psi \cup \{\alpha\}$ is consistent and using property (iv) of Lemma 2.19 it is a subset of some member, let it be Φ_0, of $\mathsf{MC}(\Delta \cup \{\alpha\})$ and the set $\Phi_0 - (\Psi \cup \{\alpha\})$ must be non-empty, because of (1) and (2). But, this is impossible because of Lemma 2.19 (i). Therefore $\Psi \cup \{\alpha\} \vdash \beta$ and $\Psi \vdash \alpha \rightarrow \beta$. Thus $\Delta \vdash_\forall \alpha \rightarrow \beta$.

For F, assume $\Delta \cup \{\alpha\} \vdash_F \beta$. Then, since $\mathsf{FREE}(\Delta \cup \{\alpha\}) \vdash \beta$ and $\mathsf{FREE}(\Delta \cup \{\alpha\}) \subseteq \mathsf{FREE}(\Delta) \cup \{\alpha\}$ (Lemma 2.20), it follows, using classical monotonicity, that $\mathsf{FREE}(\Delta) \cup \{\alpha\} \vdash \beta$. This implies $\mathsf{FREE}(\Delta) \vdash \alpha \rightarrow \beta$. The proof for F, also covers the cases for $\mathsf{U}, \mathsf{RU}, \mathsf{B}$. The cases for n, T are simple. $\Box$

Proposition 4.20 *Deduction fails for all the* x*-consequence relations, except for the* n*-consequence relation.*

Proof of 4.20 Take $\Delta = \{\neg\alpha\}$, then $\Delta \vdash_x \neg\alpha$, hence $\Delta \vdash_x \alpha \rightarrow \beta$. But $\Delta \cup \{\alpha\} \not\vdash_x \beta$. $\Box$

Proposition 4.21 *Or fails for the* $\exists, \mathsf{R}, \mathsf{F}, \mathsf{U}, \mathsf{RU}, \mathsf{B}$*-consequence relations. Or succeeds for the* $\mathsf{n}, \forall, \mathsf{T}$*-consequence relations.*

Proof of 4.21 For $\exists, \mathsf{R}$, let $\Delta = \{(\alpha \rightarrow \gamma) \wedge \neg\delta, (\beta \rightarrow \gamma) \wedge \delta\}$. For F, let $\Delta = \{\neg\alpha \wedge \gamma, \neg\beta \wedge \gamma\}$.

For $\forall$, the proof is quite complicated. The complication is to establish the cases that need to be considered. To this end Lemma 2.17 is useful. To increase readability, we introduce a special notation, using, for instance, $\mathsf{CON}(\Delta)^{\vdash\alpha}$ to abbreviate $\{\Phi \in \mathsf{CON}(\Delta) | \Phi \vdash \alpha\}$. Using this special notation, Lemma 2.17 can be used to partition $\mathsf{MC}(\Delta \cup \{\alpha \vee \beta\})$ into two disjoint sets:

$$\mathsf{MC}(\Delta \cup \{\alpha \vee \beta\}) \;=\; \mathsf{MC}(\Delta)^{\vdash\neg\alpha\wedge\neg\beta} \;\cup\; \{\Phi \cup \{\alpha \vee \beta\} | \Phi \in \mathsf{max}(\mathsf{CON}(\Delta)^{\not\vdash\neg\alpha\wedge\neg\beta})\}$$

Similar partitions are defined for $\mathsf{MC}(\Delta \cup \{\alpha\})$ and $\mathsf{MC}(\Delta \cup \{\beta\})$. What we need to understand is how members of $\mathsf{MC}(\Delta \cup \{\alpha \vee \beta\})$ relate to members of $\mathsf{MC}(\Delta \cup \{\alpha\})$ and $\mathsf{MC}(\Delta \cup \{\beta\})$. For any member, Φ, of $\mathsf{MC}(\Delta \cup \{\alpha \vee \beta\})$ there are two main cases to consider.

(1) If $\Phi \vdash \neg\alpha \wedge \neg\beta$, then it is the case that (1.1) $\Phi \vdash \neg\alpha$, (1.2) $\Phi \vdash \neg\beta$, (1.3) $\Phi \in \mathsf{MC}(\Delta \cup \{\alpha\})$ and (1.4) $\Phi \in \mathsf{MC}(\Delta \cup \{\beta\})$. [proof outline: (1.1) and (1.2) are simple. (1.3) and (1.4) can be established using the fact that $\Phi \in \mathsf{MC}(\Delta)^{\vdash\neg\alpha\wedge\neg\beta}$ and Lemma 2.19 (ii).]

(2) If $\Phi \not\vdash \neg\alpha \wedge \neg\beta$, then a $\Psi_0 \in \mathsf{max}(\mathsf{CON}(\Delta)^{\not\vdash\neg\alpha\wedge\neg\beta})$ can be picked, such that $\Phi = \Psi_0 \cup \{\alpha \vee \beta\}$. [This is a direct consequence of Lemma 2.17.] For any witness, Ψ_0, four cases are possible: (a) $\Psi_0 \not\vdash \neg\alpha \wedge \Psi_0 \cup \{\alpha\} \in \mathsf{MC}(\Delta \cup \{\alpha\})$, (b) $\Psi_0 \not\vdash \neg\beta \wedge \Psi_0 \cup \{\beta\} \in \mathsf{MC}(\Delta \cup \{\beta\})$, (c) $\Psi_0 \vdash \neg\alpha \wedge \Psi_0 \in \mathsf{MC}(\Delta \cup \{\alpha\})$, and (d) $\Psi_0 \vdash \neg\beta \wedge \Psi_0 \in \mathsf{MC}(\Delta \cup \{\beta\})$. These cases are not disjoint, but by considering three cases: (ab), (bc) and (ad), all cases are exhausted. (The combination (cd)

is not consistent with (2).) [proof outline: That this case splitting is correct, can be verified using Lemma 2.17.]

Now, the proof of the case for $\forall$ proceeds as follows. Assume $\Delta \cup \{\alpha\} \vdash_{\forall} \gamma$ and $\Delta \cup \{\beta\} \vdash_{\forall} \gamma$, to prove $\Delta \cup \{\alpha \vee \beta\} \vdash_{\forall} \gamma$. Let $\Phi \in \mathsf{MC}(\Delta \cup \{\alpha \vee \beta\})$. Then we have two main cases to consider. (1) If $\Phi \vdash \neg\alpha \wedge \neg\beta$, then Φ is a member of both $\mathsf{MC}(\Delta \cup \{\alpha\})$ and $\mathsf{MC}(\Delta \cup \{\beta\})$, cf. (1.3) and (1.4) above, and by assumption $\Phi \vdash \gamma$. (2) If $\Phi \not\vdash \neg\alpha \wedge \neg\beta$, then we can pick some $\Psi_0 \in \mathsf{max}(\mathsf{CON}(\Delta)^{\not\vdash \neg\alpha\wedge\neg\beta})$, such that $\Phi = \Psi_0 \cup \{\alpha \vee \beta\}$. There are three subcases to consider: (ab) In this case, $\Psi_0 \cup \{\alpha\} \vdash \gamma$ and $\Psi_0 \cup \{\beta\} \vdash \gamma$ by the assumptions. Using the classical or-property we get $\Psi_0 \cup \{\alpha \vee \beta\} \vdash \gamma$, i.e. $\Phi \vdash \gamma$. (bc) In this case $\Psi_0 \in \mathsf{MC}(\Delta \cup \{\alpha\})$ and therefore, using monotonicity of classical logic, $\Phi \vdash \gamma$. (ad) Similar to (bc)-case. This exhausts all possibilities and thus $\Delta \cup \{\alpha \vee \beta\} \vdash_{\forall} \gamma$. $\Box$

In considering each of the argumentative logics, it has been of interest to find properties that distinguish them from the other members of the family of logics induced by the notion of acceptability related to A^*. To emphasize this aspect, the above results are summarized in the table in Figure 2.

The weaker conditions on $\mathsf{C}\exists$ and CR allow some of the classical axioms on the consequence relations to hold such as left logical equivalence and right weakening, but that others such as cut, the and property, and the or property fail. In contrast for the more restricted definitions for $\mathsf{C}\forall$, CU, CRU, CB and CF, monotonicity fails, but cut and the and property are preserved. A number of properties succeed for all these consequence relations such as left logical equivalence, right weakening, and conditionalization, whereas some properties fail for all (except the Cn-consequence relation) such as supraclassicality and deduction.

Note how even though a property may fail for a consequence relation, further restrictions on the acceptability of arguments may cause the property to hold. For example, the properties of and, cautious monotony, and cut fail for CR, but succeed for $\mathsf{C}\forall$. For the or property, increasing the restrictions on acceptability causes failure for $\mathsf{C}\exists$ and CR, success for $\mathsf{C}\forall$, and then failure for CU.

5 Iterating consequence closure functions

It is of interest to consider iterating consequence closures for handling inconsistent information, since at least for some functions, it might be possible to repeatedly apply the consequence closure to improve the quality of the resulting inferences. In other words, a consequence closure might remove problematical data, and hence if repeated, might lead to more acceptable inferences. We explore this idea in the following subsections.

Properties	Cn	C∃	CR	C∀	CU	CT	Proposition
Supraclassicality	•	◦	◦	◦	◦	◦	4.9
Reflexivity	•	◦	◦	◦	◦	◦	4.10
Left logical equivalence	•	•	•	•	•	•	4.11
Right weakening	•	•	•	•	•	•	4.12
And	•	◦	◦	•	•	•	4.13
Rational monotony	•	•	◦	◦	◦	•	4.14
Cautious monotony	•	•	◦	•	•	•	4.15
Monotonicity	•	•	◦	◦	◦	•	4.16
Cut	•	◦	◦	•	•	•	4.17
Consistency preservation	•	•	•	•	•	•	4.18
Conditionalization	•	•	•	•	•	•	4.19
Deduction	•	◦	◦	◦	◦	◦	4.20
Or	•	◦	◦	•	◦	•	4.21

Figure 2: Summary of properties. Symbols: •, for success and ◦, for failure

Definition 5.1 *An i-iterated* x*-consequence closure is denoted Cx^i and defined as follows,*

$$\begin{array}{lcl} \mathsf{Cx}^1(\Delta) & = & \mathsf{Cx}(\Delta) \\ \mathsf{Cx}^{n+1}(\Delta) & = & \mathsf{Cx}(\mathsf{Cx}^n(\Delta)) \\ \mathsf{Cx}^*(\Delta) & = & \mathsf{Cx}^n(\Delta), \quad \textit{iff } \forall m \geq n \cdot \mathsf{Cx}^m(\Delta) = \mathsf{Cx}^n(\Delta) \end{array}$$

5.1 Properties of C∃

Proposition 5.2 *Let Δ be such that $|\mathsf{MC}(\Delta)| > 1$. Then:*

(i) $\mathsf{C}\exists(\Delta)$ *is inconsistent.*

(ii) $\mathsf{C}\exists^2(\Delta) = \{\phi \in \mathsf{Cn}(\Delta) | \phi \not\vdash \bot\}$.

(iii) $\mathsf{C}\exists^*(\Delta) = \mathsf{C}\exists^2(\Delta) = \{\phi \in \mathsf{Cn}(\Delta) | \phi \not\vdash \bot\}$.

Proof 5.3 For part (i), obvious. For part (ii), use Lemma 2.14 to pick $\phi_0, \neg\phi_0 \in \mathsf{C}\exists(\Delta)$. By or-introduction $\phi_0 \vee \gamma \in \mathsf{CE}(\Delta)$ for any $\gamma \in \mathcal{L}$. By definition of C∃, if $\neg\phi_0, \phi_0 \vee \gamma$ is consistent, then $\gamma \in \mathsf{C}\exists^2(\Delta)$. For part (iii), consider part (ii) and the axiom of reflexivity holding for the ∃-consequence relation. □

5.2 Properties of CR

Proposition 5.4 $\forall \Pi \in \mathsf{MC}(\Delta), \phi \in \mathsf{CR}(\Delta) \cdot \Pi \cup \phi \not\vdash \bot$

Proof of 5.4 If $\phi \in \mathsf{CR}(\Delta)$, then for all $\Pi \in \mathsf{CON}(\Delta)$, $\Pi \nvdash \neg\phi$ by definition of CR. Therefore $\Pi \cup \phi$ is consistent. □

Proposition 5.5 *For some Δ it is the case that $\phi, \psi \in \mathsf{CR}(\Delta)$ and $\Pi \in \mathsf{CON}(\Delta)$, but $\Pi \cup \{\phi, \psi\}$ is inconsistent.*

Proof of 5.5 Let $\Delta = \{((\neg\alpha \vee \neg\beta) \wedge \sigma \wedge \gamma), (\alpha \wedge \sigma \wedge \neg\gamma), (\beta \wedge \neg\sigma \wedge \gamma)\}$. Then $\alpha, \beta \in \mathsf{CR}(\Delta)$, $\{(\neg\alpha \vee \neg\beta) \wedge \sigma \wedge \gamma\} \in \mathsf{MC}(\Delta)$ and $\{(\neg\alpha \vee \neg\beta) \wedge \sigma \wedge \gamma\} \cup \{\alpha, \beta\}$ is inconsistent. □

Lemma 5.6 *Let $\alpha, \neg\alpha \in \mathsf{C}\exists(\Delta)$ and γ be $\exists$-undecided. Then $\alpha, \neg\alpha \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$ and also $\gamma, \neg\gamma \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$.*

Proof of 5.6 From the definition of the consequence relation we have that:

$$\alpha \vee \gamma, \neg\alpha \vee \gamma, \alpha \vee \neg\gamma, \neg\alpha \vee \neg\gamma \in \mathsf{CR}(\Delta)$$

because

$$\neg\alpha \wedge \neg\gamma, \alpha \wedge \neg\gamma, \neg\alpha \wedge \gamma, \alpha \wedge \gamma \notin \mathsf{C}\exists(\Delta).$$

Any two of the above elements in $\mathsf{CR}(\Delta)$ are consistent subsets. Therefore we get:

$$\alpha, \neg\alpha, \gamma, \neg\gamma \in \mathsf{C}\exists(\mathsf{CR}(\Delta)).$$

From this both conclusions are immediate. □

Lemma 5.7 *Let Δ be such that $|\mathsf{MC}(\Delta)| > 1$ and some sentence is $\exists$-undecided by Δ. Then for any non-tautological $\exists$-consequence, ϕ, of Δ,$\neg\gamma \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$.*

Proof of 5.7 Pick $\alpha_0, \neg\alpha_0$ according to Lemma 2.14, and so $\Delta \vdash_\exists \alpha$, and $\Delta \vdash_\exists \neg\alpha$, and let $\gamma_0, \neg\gamma_0$ be two sentences that are $\exists$-undecided by Δ. Either $\neg\phi \notin \mathsf{C}\exists(\Delta)$ or we can use a technique similar to the one employed in Lemma 5.6 and the result will follow. In the former case, the following cases must be considered, investigating all possible logical interrelationships of interest between the sentences we use to construct a $\neg\phi$.

($\phi \vdash \alpha_0$): In this case contraposition give $\neg\alpha_0 \vdash \neg\phi$. Using Lemma 5.6 gives $\neg\alpha_0$ and Lemma 4.7 ensures that $\neg\alpha_0$ gives $\neg\phi$.

($\phi \vdash \neg\alpha_0$): As above, using $\alpha_0 \vdash \neg\phi$ and α_0 instead.

($\phi \nvdash \alpha_0$) and ($\phi \nvdash \neg\alpha_0$): This case has three subcases. (These are necessary in order to ensure that constraints between the sentences we use in the construction of $\neg\phi$ cannot spoil the consistency of the set of these sentences. The possible constraints demark the different cases, but the constraints are themselves unnecessary for producing the $\neg\phi$!)

$(\alpha_0 \vdash \phi)$ or $(\neg\gamma_0 \vdash \phi)$: The following sentences are R-consequences of Δ: (i) $\alpha_0 \vee \neg\gamma_0 \vee \neg\phi$, (ii) $\neg\alpha_0 \vee \gamma_0$, (iii) $\alpha_0 \vee \gamma_0$ and (iv) $\neg\alpha_0 \vee \neg\gamma_0$. Since γ_0 is $\exists$-undecided by Δ there is no way these sentences can be rebutted by other sentences that can be $\exists$-derived from Δ. The set consisting of (i)–(iv) is consistent and has the unique 'model' $\{\neg\alpha_0, \gamma_0, \neg\phi\}$. This model also satisfies each of the constraints $(\alpha_0 \vdash \phi)$ or $(\neg\gamma_0 \vdash \phi)$. The sentences (i)-(iv) imply $\neg\phi$.

$(\neg\alpha_0 \vdash \phi)$ or $(\gamma_0 \vdash \phi)$: Using a similar line of argument as in the previous subcase, we can construct the sentences: (i) $\neg\alpha_0 \vee \gamma_0 \vee \neg\phi$, (ii) $\neg\alpha_0 \vee \neg\gamma_0$, (iii) $\alpha_0 \vee \neg\gamma_0$ and (iv) $\alpha_0 \vee \gamma_0$, which are R-consequences of Δ. The set consisting of (i)–(iv) is consistent and has the unique 'model' $\{\alpha_0, \neg\gamma_0, \neg\phi\}$. This model also satisfies each of the constraints $(\neg\alpha_0 \vdash \phi)$ or $(\gamma_0 \vdash \phi)$. The sentences (i)-(iv) imply $\neg\phi$.

Neither of these: In this case there are no extra constraints and either set of sentences (i)–(iv) from the previous two subcases will do the job.

With these we have exhausted all possibilities, and the result follows. □

Example 5.8 [Related to Lemma 5.7]
Let $\Delta = \{\alpha, \neg\alpha\}$. Suppose α is the only non-logical symbol in the language. Then $\mathsf{CR}(\Delta) = \mathsf{CR}(\emptyset)$. Suppose instead that α and β are the two only (and non-identical) non-logical symbols in the language. Then $\mathsf{CR}(\Delta)$ includes $(\alpha \vee \beta), (\neg\alpha \vee \beta), \ldots$ in addition to the set of tautologies. □

Lemma 5.9
If Δ does not $\exists$-decide every formula in $\mathcal{L}$, then $|\mathsf{MC}(\Delta)| > 1$ implies $\mathsf{CR}^2(\Delta) = \mathsf{Cn}(\emptyset)$.

Proof of 5.9 Suppose $\phi \in \mathsf{CR}(\mathsf{CR}(\Delta))$ and ϕ is not tautological. By Lemma 5.6 ϕ cannot be $\exists$-undecided, because this would contradict the first assumption. Therefore we have three mutually exclusive cases to consider:

(i) $\phi \in \mathsf{CR}(\Delta)$: By Lemma 5.7 $\neg\phi \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$ contradicting the first assumption.

(ii) $\neg\phi \in \mathsf{CR}(\Delta)$: By Lemma 5.7 $\phi \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$ and (ii) gives $\neg\phi \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$. This contradicts the first assumption.

(iii) $\phi \in \mathsf{C}\exists(\Delta)$ and $\neg\phi \in \mathsf{C}\exists(\Delta)$: In this case Lemma 5.6 gives $\phi \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$ and $\neg\phi \in \mathsf{C}\exists(\mathsf{CR}(\Delta))$. Again this contradicts the first assumption.

Therefore, $\phi \notin \mathsf{CR}(\mathsf{CR}(\Delta))$ or ϕ is tautological. □

Proposition 5.10 *Let Δ be such that $|\mathsf{MC}(\Delta)| > 1$ and some sentence is $\exists$-undecided by Δ. Then*

(i) $\mathsf{CR}(\Delta)$ *is inconsistent.*

(ii) $\mathsf{CR}(\Delta) \subset \mathsf{Cn}(\mathsf{CR}(\Delta))$.

(iii) $\mathsf{CR}^2(\Delta) = \mathsf{Cn}(\emptyset)$.

(iv) $\mathsf{CR}^*(\Delta) = \mathsf{CR}^2(\Delta) = \mathsf{Cn}(\emptyset)$.

Proof of 5.10

(i) Follows immediately from Lemma 5.6.

(ii) Case $\neq$: $\mathsf{CR}(\Delta) \vdash \bot$, but $\bot \notin \mathsf{CR}(\Delta)$. Case $\subset$: follows from the monotonicity of Cn.

(iii) This is the result of Lemma 5.9.

(iv) Immediate from (iii).

□

5.3 Properties of $\mathsf{C}\forall$

Proposition 5.11 *Let* Δ *be such that* $|\mathsf{MC}(\Delta)| > 1$. *Then*

(i) $\mathsf{C}\forall(\Delta)$ *is consistent for any* Δ.

(ii) $\mathsf{C}\forall(\Delta) = \mathsf{Cn}(\mathsf{C}\forall(\Delta))$.

(iii) $\mathsf{C}\forall^*(\Delta) = \mathsf{C}\forall(\Delta)$ *for any* Δ.

Proof of 5.11 (i) Assume that $\mathsf{C}\forall(\Delta)$ is inconsistent. Then each of the members of $\mathsf{MC}(\Delta)$ must be inconsistent. An absurdity. (ii) and (iii) follows immediately. □

This immediately gives analogous results for CU, CRU, CB, CF and CT, because they are also consistent.

Example 5.12 Let $\Delta = \{((\neg\alpha \vee \neg\beta) \wedge \sigma \wedge \gamma), (\alpha \wedge \sigma \wedge \neg\gamma), (\beta \wedge \neg\sigma \wedge \gamma)\}$. Then $\mathsf{C}\forall(\Delta) = \mathsf{Cn}(\emptyset)$. □

5.4 Utility of iterating consequence closure functions

Iteration of the $\mathsf{C}\exists$ function removes relatively little information from Δ, in general. Two iterations then introduces many "trivial" formulae into the closure, and furthermore, the closure stabilizes. It is therefore clear that iterating the $\mathsf{C}\exists$ function does not improve the information content of the closure.

For the R-consequence closure, some inconsistent information is removed, but Proposition 5.5 shows that not all inconsistent information is necessarily removed. Furthermore, iterating this closure function is not useful, as is demonstrated by Lemma 5.9.

Finally, even though the $\forall$-consequence closure does remove inconsistent information from the database, it stabilizes after only one iteration. The same comment applies to iterating the U, RU, B, F, T-consequence closures.

6 Prime implicants

A database can be represented by its set of prime implicants. The following definition of prime implicants is a generalization of a similar one of Benferhat et al (1993).

Definition 6.1 *A prime* x*-implicant is a sentence* $\phi \in \mathsf{Cx}(\Delta)$ *such that:*

$$\forall \psi \in \mathsf{Cx}(\Delta) - \mathsf{Cn}(\phi) \cdot \psi \not\vdash \phi$$

The set of prime implicants is defined as:

$$\mathsf{PIx}(\Delta) = \{\phi \in \mathsf{Cx}(\Delta) | \forall \psi \in \mathsf{Cx}(\Delta) - \mathsf{Cn}(\phi) \cdot \psi \not\vdash \phi\}.$$

Example 6.2 [Prime n-implicants] Let $\Delta = \{\alpha \wedge \alpha, \alpha, \beta, \alpha \vee \beta\}$. Then $\mathsf{PIn}(\Delta) = \{\alpha \wedge \alpha, \alpha, \beta\}$. Two of these prime n-implicants are logically equivalent and a unique representative could be chosen to represent this equivalence set (but we will not pursue this here). □

A desirable general result would be the ability to represent any database finitely by its set of prime implicants w.r.t. some consequence relation. Unfortunately, this is not possible since in some cases the size of the set of prime implicants is similar to the size of the language, as shown in Example 6.3.

Example 6.3 Consider the database: $\Delta = \{\neg\alpha, \alpha\}$ and suppose the language is countable infinite. Then for each β in the language, such that $\beta \not\vdash \alpha$ the formula $\alpha \vee \beta$ will be a prime R-implicant, and similarly for each γ such that $\gamma \not\vdash \neg\alpha$ the formula $\neg\alpha \vee \gamma$ will be a prime R-implicant. Therefore the set of prime R-implicants for Δ is infinite. □

Proposition 6.4 *Let* Δ *be finite. Then* $\mathsf{Cx}(\Delta) = \bigcup\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Plx}(\Delta)\}$.

Proof of 6.4 Using Lemma 4.7 we get

$$\mathsf{Cx}(\Delta) = \bigcup\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Cx}(\Delta)\}$$

Using the fact that any finite set of sets has maximal elements under the $\subseteq$-relation and other standard set-properties, we get, using max as an operator picking maximal elements:

$$\begin{aligned} & \bigcup\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Cx}(\Delta)\} \\ = & \bigcup \mathsf{max}\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Cx}(\Delta)\} \\ = & \bigcup\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Cx}(\Delta) \wedge \forall\psi \in \mathsf{Cx}(\Delta) \cdot \psi \vdash \phi \rightarrow \phi \vdash \psi\} \\ = & \bigcup\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Cx}(\Delta) \wedge \forall\psi \in \mathsf{Cx}(\Delta) \wedge \psi \notin \mathsf{Cn}(\phi) \cdot \psi \nvdash \phi\} \\ = & \bigcup\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Cx}(\Delta) \wedge \forall\psi \in \mathsf{Cx}(\Delta) - \mathsf{Cn}(\phi) \cdot \psi \nvdash \phi\} \\ = & \bigcup\{\mathsf{Cn}(\phi)|\phi \in \mathsf{Plx}(\Delta)\} \end{aligned}$$

(Infinite databases might not have any prime implicants. For instance let α_i, $i \in \mathsf{Nat}$, enumerate distinct atomic propositional variables, then $\Delta = \{\bigwedge_{i\leq n} \alpha_i | n \in \mathsf{Nat}\}$ does not have any prime implicants. The present proof would fail in this case, because this Δ does not have any maximal elements.) □

7 Perspectives

Through the definition of A^*, we have shown how arguments of varying degrees of acceptability can be identified in the context of an inconsistent database. For example, arguments in the AF argument set are more acceptable than arguments in the $\mathsf{A}\forall$ or $\mathsf{A}\exists$ argument sets. If you have a classical database, and all you know is that it is inconsistent, then the argumentative structure can be used to make distinctions between different arguments. This aspect of the framework distinguishes it from other non-monotonic logics and from truth-maintenances systems.

Even though this work is useful as a first understanding of how to handle and use inconsistent information, it leaves open questions of other ways to select acceptable or "preferred" inferences from inconsistent information or ways to select acceptable, or "preferred", premises. It also leaves open how an argumentative structure can be defined for defeasible information. In particular, how the difference between "hard" and "defeasible" information can be expressed in an appropriate notion of acceptability.

Another important issue is how an argumentative structure can be designed to allow for explicit priorities to resolve conflicts. Priorities can be represented by labels on the formulae in the language. Such labels can be used to facilitate selection of preferred subsets of the database, and if propagated by the proof rules, they can be used to facilitate selection of preferred inferences. The use of labelling has been motivated by the approach of labelled deductive systems (Gabbay 1991), and has been developed to capture the general notion of priorities in non-monotonic reasoning (Hunter 1992). Priorities allow for more sophisticated, and arguably more appropriate handling of inconsistent and default information. Initial investigations indicate that priorities should provide interesting developments of argumentative structures and argumentative logics (Elvang-Gøransson et al 1993).

There are also a number of other argument-based systems that have been proposed, including by Vreeswijk (1991), Prakken (1991), and Simari (1992). These differ from our work in that they focus on defeasible reasoning: They incorporate defeasible, or default, connectives into their languages, together with associated machinery.

Another approach to acceptability of arguments is Dung (1993). This approach assumes a set of arguments, and a binary "attacks" relation between pairs of subsets of arguments. A hierarchy of arguments is then defined in terms of the relative attacks 'for' and 'against' each argument in each subset of arguments. In this way, for example, the plausibility of an argument could be defended by another argument in its subset. Whilst there are significant differences with our approach, a comparison would be a worthwhile goal.

In conclusion, we see the concept of an argumentative structure, with the two notions of argument and acceptability, as a convenient framework for expressing adequate practical reasoning tools. Although, we use simple definitions of arguments and acceptability, these concepts carry many possibilities for further refinement. It remains to be seen whether there is a general taxonomy of argumentative structures (Pinkas and Loui 1992) and universal properties of the logics that they induce.

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