Appendix

Proposition 5. *Algorithm 1 terminates.*

Proof. To ensure termination, we show that it is not possible for the `GetModels` method to be involved in an infinite recursion. For the first four cases, the formula is decomposed into strict subformula according to the Boolean connective and each of these subformulae is a recursive call to the method. For instance, for the first case, the call for $\psi_1 \wedge \psi_2$ involves recursive calls for ψ_1 and for ψ_2. For the fifth case, the call is for an epistemic atom, and this constitutes a base case as it does not involve recursive calls. For the sixth case, either the combination set is empty set, in which this constitutes a base case as it does not involve recursive calls, or the formula is decomposed into a finite number of epistemic atoms each of which is recursive call to the method. Since each formula can only be decomposed into a finite number of subformulae, and each call decomposes each formulae into a finite number of strict subformulae each for a recursive call, or is a base case, the algorithm terminates.

Proposition 2. *If T_1 and T_2 are decomposition trees for ϕ, then* $\mathsf{Leaves}(T_1) = \mathsf{Leaves}(T_2)$.

Proof. Since the propositional rules split the epistemic formulae into epistemic atoms based on the Boolean connectives, the epistemic atoms that appear in T_1 are identical to those that appear in T_2. For an epistemic formula that is composed of three or more conjuncts (respectively disjuncts), there is a choice for the decomposition rule to split the formula. For example, $\psi_1 \wedge \psi_2 \wedge \psi_3$ can be split as ψ_1 and $\psi_2 \wedge \psi_3$ or as $\psi_1 \wedge \psi_2$ and ψ_3. But this choice does not affect the epistemic atoms that eventually appear in the branches. Furthermore, for each epistemic atom, each application of a non-propositional decompositions rule is identical for both trees. Therefore, $\mathsf{Leaves}(T_1) = \mathsf{Leaves}(T_2)$.

Proposition 3. *If T_1 and T_2 are decomposition trees for ϕ, and the root of T_1 (respectively T_2) is n_1 (respectively n_2), then* $\mathsf{Models}(n_1) = \mathsf{Models}(n_2)$.

Proof. Proof by induction. Let $\mathsf{Subtree}(n, T)$ denote the subtree of T that is rooted at n.

First, consider the subtree root at n_1 (respectively n_2) where there is just one child n'_1 (respectively n'_2) and that is a leaf. So n'_1 (respectively n'_2) is obtained by the term decomposition rule or is an instance of the operational rule producing the empty set. So if $\mathsf{Models}(n'_1) = \mathsf{Models}(n'_2)$, then $\mathsf{Models}(n_1) = \mathsf{Models}(n_2)$.

Second, consider the subtree root at n_1 (respectively n_2) where there is just one child n'_1 (respectively n'_2) and that is not a leaf and has been obtained by the application of the operational decomposition rule that does not produce the empty set. So $\mathsf{Models}(n_1) = \mathsf{Models}(n'_1)$ and $\mathsf{Models}(n_2) = \mathsf{Models}(n'_2)$. Hence, if $\mathsf{Models}(n'_1) = \mathsf{Models}(n'_2)$, then $\mathsf{Models}(n_1) = \mathsf{Models}(n_2)$.

Third, consider the subtree root at n_1 (respectively n_2) where there is just one child n'_1 (respectively n'_2) and that is not a leaf and has been obtained by the application of the negation propositional decomposition rule. So $\mathsf{Models}(n_1) = \mathsf{Dist}(\mathcal{G}, \pi) \setminus \mathsf{Models}(n'_1)$ and $\mathsf{Models}(n_2) = \mathsf{Dist}(\mathcal{G}, \pi) \setminus \mathsf{Models}(n'_2)$. Hence, if $\mathsf{Models}(n'_1) = \mathsf{Models}(n'_2)$, then $\mathsf{Models}(n_1) = \mathsf{Models}(n_2)$.

Fourth, consider the subtree root at n_1 (respectively n_2) where there are two children n'_1 and n''_1 (respectively n'_2 and n''_2) and that are not leaves and has been obtained by the application of the implication propositional decomposition rule. So $\mathsf{Models}(n_1) = \mathsf{Models}(n'_1) \cup \mathsf{Models}(n''_1)$ and $\mathsf{Models}(n_2) = \mathsf{Models}(n'_2) \cup \mathsf{Models}(n''_2)$. Hence, if $\mathsf{Models}(n'_1) = \mathsf{Models}(n'_2)$, and $\mathsf{Models}(n''_1) = \mathsf{Models}(n''_2)$, then $\mathsf{Models}(n_1) = \mathsf{Models}(n_2)$.

Fifth, consider the subtree root at n_1 (respectively n_2) where there are two children n'_1 and n''_1 (respectively n'_2 and n''_2) that are not leaves and has been obtained by the application of the disjunction propositional decomposition rule on a formula with more than two disjuncts, then the formulae at the children may be different in the two trees. For example, $\psi_1 \wedge \psi_2 \wedge \psi_3$ can be split as ψ_1 and $\psi_2 \wedge \psi_3$ or as $\psi_1 \wedge \psi_2$ and ψ_3. But in both cases, there is a node in each tree for ψ_1, ψ_2 and ψ_3. So by applying the same argument as for the fourth case above, $\mathsf{Models}(n_1) = \mathsf{Models}(n_2)$.

Sixth, consider the subtree root at n_1 (respectively n_2) where there are two children n'_1 and n''_1 (respectively n'_2 and n''_2) that are not leaves and has been obtained by the application of the conjunction propositional decomposition rule on a formula with more than two disjuncts, then we can apply the same argument as in the fifth case above but with intersection rather union used to obtain the models.

Therefore, for each epistemic atom ψ, if ψ is at n_1 in T_1 and n_2 in T_2, then $\mathsf{Models}(n_1) = \mathsf{Models}(n_2)$.

Proposition 5. *For the conjunction propositional decompositional rule, with condition ϕ, and consequent $\psi_1 \mid \psi_2$, the following holds for a model P: $P \in \mathsf{Sat}(\phi, \Pi)$ iff $P \in \mathsf{Sat}(\psi_1, \Pi)$ and $P \in \mathsf{Sat}(\psi_2, \Pi)$.*

Proof. Consider the condition $\phi = \phi_1 \wedge \phi_2$. Therefore for all models P, $P \in \mathsf{Sat}(\phi_1 \wedge \phi_2, \Pi)$ iff $P \in \mathsf{Sat}(\phi_1, \Pi)$ and $P \in \mathsf{Sat}(\phi_2, \Pi)$.

Proposition 6. *For the disjunction and implication propositional decomposition rules, with condition ϕ, and consequent $\psi_1 \mid \psi_2$, the following holds for a model P: $P \in \mathsf{Sat}(\phi, \Pi)$ iff $P \in \mathsf{Sat}(\psi_1, \Pi)$ or $P \in \mathsf{Sat}(\psi_2, \Pi)$.*

Proof. Consider the disjunction propositional decomposition rule. So the condition $\phi = \phi_1 \vee \phi_2$. Therefore for all models P, $P \in \mathsf{Sat}(\phi_1 \vee \phi_2, \Pi)$ iff $P \in \mathsf{Sat}(\phi_1, \Pi)$ or $P \in \mathsf{Sat}(\phi_2, \Pi)$. We can make an analogous argument for the implication propositional decomposition rule.

Proposition 7. *For the negation propositional decomposition rules, with condition $\neg\phi$, and consequent ϕ, the following holds for a model P: $P \in \mathsf{Sat}(\neg\phi, \Pi)$ iff $P \in \mathsf{Dist}(\mathcal{G}, \phi) \setminus \mathsf{Sat}(\phi, \Pi)$.*

Proof. Holds straightforwardly from the properties of Sat.

Proposition 8. *For the operational decomposition rule with condition* $p(\alpha_1) *_1 \dots *_n p(\alpha_{n+1}) \# x$*, and consequent* $\bigvee_{(v_1,\dots,v_{n+1}) \in \Pi_{\#}^{x,(*_1,\dots,*_n)}} (p(\alpha_1) = v_1 \wedge \dots \wedge p(\alpha_{n+1}) = v_{n+1})$ *and a model* P*, it holds that* $P \in \mathsf{Sat}(p(\alpha_1) *_1 \dots *_n p(\alpha_{n+1}) \# x, \Pi)$ *iff*

$$P \in \mathsf{Sat}\Big(\bigvee_{(v_1,\dots,v_{n+1}) \in \Pi_{\#}^{x,(*_1,\dots,*_n)}} (p(\alpha_1) = v_1 \wedge \dots \wedge p(\alpha_{n+1}) = v_{n+1}), \Pi\Big)$$

Proof. Let $\psi : p(\alpha_1) *_1 \dots *_n p(\alpha_{n+1})$ and $Q = (*_1, \dots, *_n)$. We first consider $\#$ being $>$. Given the assumed consequent, it holds that $\Pi_{\#}^{x,Q} \neq \varnothing$. For every $P' \in \mathsf{Sat}(\psi > x, \Pi)$, $P'(\alpha_1) *_1 \dots *_n P'(\alpha_{n+1}) > x$. Consequently, $(P'(\alpha_1), \dots, P'(\alpha_{n+1})) \in \Pi_{>}^{x,Q}$. We can therefore show that $\mathsf{Sat}(\psi > x, \Pi) \subseteq \mathsf{Sat}(\bigvee_{(v_1,\dots,v_{n+1}) \in \Pi_{>}^{x,Q}} (p(\alpha_1) = v_1 \wedge p(\alpha_2) = v_2 \wedge \dots \wedge p(\alpha_{n+1}) = v_{n+1}), \Pi)$.

Let now $P' \in \mathsf{Sat}(\bigvee_{(v_1,\dots,v_{n+1}) \in \Pi_{>}^{x,Q}} (p(\alpha_1) = v_1 \wedge \dots \wedge p(\alpha_{n+1}) = v_{n+1}), \Pi)$. Based on the properties of Sat, it means that there is $(v_1, \dots, v_{n+1}) \in \Pi_{>}^{x,Q}$ s.t. $P' \in \mathsf{Sat}((p(\alpha_1) = v_1 \wedge \dots \wedge p(\alpha_{n+1}) = v_{n+1}), \Pi)$. Since $v_1 *_1 v_2 *_2 \dots *_n v_{n+1} > x$, then $P'(\alpha_1) *_1 \dots *_n P'(\alpha_{n+1}) > x$. Hence, $P' \in \mathsf{Sat}(\psi > x, \Pi)$, and we can show that $\mathsf{Sat}(\bigvee_{(v_1,\dots,v_{n+1}) \in \Pi_{>}^{x,Q}} (p(\alpha_1) = v_1 \wedge \dots \wedge p(\alpha_{n+1}) = v_{b+1}), \Pi) \subseteq \mathsf{Sat}(\psi > x, \Pi)$.

Given the previous result, $\mathsf{Sat}(\psi > x, \Pi) = \mathsf{Sat}(\bigvee_{(v_1,\dots,v_{n+1}) \in \Pi_{>}^{x,Q}} (p(\alpha_1) = v_1 \wedge \dots \wedge p(\alpha_{n+1}) = v_{n+1}), \Pi)$. The results for other operators can be obtained in a similar fashion.

For the next proof, we use the following property which is an excerpt from a more general proposition from [6]:

Proposition 9. *Let* Π *be a reasonable restricted value set,* $x \in \Pi$ *a value,* $\# \in \{=, \neq, \geq, \leq, >, <\}$ *an inequality, and* $(*_1, \dots, *_k)$ *a sequence of operators where* $*_i \in \{+, -\}$ *and* $k \geq 0$*. Let* $max(\Pi)$ *denote the maximal value of* Π*. The following hold:*

- $\Pi_{\#}^{x} = \varnothing$ *if and only if:*
  1. $\#$ *is* $>$ *and* $x = max(\Pi)$*, or*
  2. $\#$ *is* $<$ *and* $x = 0$*.*
- $\Pi_{\#}^{x,(*_1,\dots,*_k)} = \varnothing$ *if and only if:*
  1. $k = 0$ *and* $\Pi_{\#}^{x} = \varnothing$*, or*
  2. $k > 0$*,* $\#$ *is* $>$*,* $x = max(\Pi)$ *and for no* $*_i$*,* $*_i = +$*, or*
  3. $k > 0$*,* $\#$ *is* $<$*,* $x = 0$ *and for no* $*_i$*,* $*_i = -$*.*

Proposition 10. *For the operational decomposition rule with condition* $p(\alpha_1) *_1 \dots *_n p(\alpha_{n+1}) \# x$*, and consequent* $\varnothing$*, it holds that* $\mathsf{Sat}(p(\alpha_1) *_1 \dots *_n p(\alpha_{n+1}) \# x, \Pi) = \varnothing$*.*

Proof. We first consider $\#$ being $>$. Given the assumptions on the consequent, it holds that $\Pi_{>}^{x,(*_1,\dots,*_n)} = \varnothing$. Based on Proposition 9 it holds that $x = 1$ and either $n = 0$ or for no $*_i$, $*_i = +$. If $n = 0$, then the condition is $p(\alpha_1) > 1$ and it is easy to see that $\mathsf{Sat}(p(\alpha_1) > 1, \Pi) = \varnothing$. If for every $*_i$, $*_i = -$, then based on the fact that probabilities belong to the unit interval, $p(\alpha_1) - p(\alpha_2) - \ldots - p(\alpha_{n+1}) \geq p(\alpha_1)$. Hence, if $\mathsf{Sat}(p(\alpha_1) > 1, \Pi) = \emptyset$, then $\mathsf{Sat}(p(\alpha_1) - p(\alpha_2) - \ldots - p(\alpha_{n+1}) > 1, \Pi) = \emptyset$ as well. Hence, for # => and the assumptions stated in the proposition, it holds that $\mathsf{Sat}(p(\alpha_1) *_1 \ldots *_n p(\alpha_{n+1}) \# x, \Pi) = \emptyset$. Similar analysis can be carried out for remaining (in)equalities.

Proposition 4. *If T is a decomposition tree for epistemic formula ϕ, and the root of the tree is node n, then $\mathsf{Sat}(\phi, \Pi) = \mathsf{Models}(n)$.*

Proof. Let $\mathsf{Formula}(n)$ denote the formula epistemic formula assigned to non-leaf node n. For each non-leaf node n with one child (respectively two children) Propositions 7, 8 and 10 (respectively Propositions 5 and 6) imply $\mathsf{Models}(n) = \mathsf{Sat}(\mathsf{Formula}(n), \Pi)$. Therefore, by induction on the tree, $\mathsf{Sat}(\phi, \pi) = \mathsf{Models}(n)$ when $\mathsf{Formula}(n) = \phi$ for the root n.

Lemma 1. *If Π is a reasonable restricted value set, then there is an integer n s.t. Π is compatible with n.*

Proof. First, by the properties of reasonable restricted value sets, we note that $\{0, 1\} \subseteq \Pi$. Let Π contain $n+1$ values and let $v_0, v_1, \ldots, v_n$ be these values in ascending order. So $v_0 = 0$ and $v_n = 1$ since these are always in Π.

First, we show that Π is uniformly graduated (i.e. the gap between each pair of numbers is the same in Π). For each v_i in the sequence, where $i < n$ and v_{i+1} is the next item in the sequence after v_i, and each v_j in the sequence, where $j < n$ and v_{j+1} is the next item in the sequence after v_j we have $v_{i+1} - v_i = v_{j+1} - v_j$. To show this, assume $v_{i+1} - v_i \neq v_{j+1} - v_j$. Without loss of generality, suppose $v_{i+1} - v_i < v_{j+1} - v_j$. Let $d_i = v_{i+1} - v_i$. By the definition of a restricted value set, d_i is also in Π. Also by the definition of a restricted value set, $v_{j+1} - d_i$ is in Π, Since, $v_{j+1} - d_i$ is bigger than v_j, v_{j+1} cannot be the next item in the sequence after v_i. From this contradiction, we have shown that uniform graduation holds (i.e. $v_{i+1} - v_i = v_{j+1} - v_j$).

Next, consider the sequence $v_0, v_1, \ldots, v_n$. The gap between each value and the next is v_1 (i.e. $v_{i+1} - v_i = v_1$). Therefore, we have $i = v_i / v_1$ for each v_i in the sequence. If we let $k = 1/v_1$, then $f(v_i) = k \cdot v_i$ is the integer i, and f is a bijection from Π to the integers $\{0, \ldots, n\}$. Furthermore, based on the distance uniformity and the properties of Π, we can show that k is a natural number. So Π is compatible with n.

Proposition 6. *Let Π be compatible with integer n. The cardinality of the set of probability distributions for Π and $\mathcal{G}$ is given by the following binomial coefficient (using the stars and bars method [5]) where $k = 2^{|\mathsf{Nodes}(\mathcal{G})|}$*

$$\binom{n+k-1}{n} = \frac{(n+k-1)!}{(k-1)!n!}$$

Proof. Each probability distribution has an assignment for each subset of the set of arguments $\mathsf{Nodes}(\mathcal{G})$. Hence, each probability distribution can be represented by a k-tuple of values from Π (one value per subset of $\mathsf{Nodes}(\mathcal{G})$).

Obviously, each k-tuple of values from Π sums to 1. Let the set of these k-tuples be denoted $\mathsf{Tuples}(k, \Pi)$. From Lemma 1, we assume that Π is compatible with integer n. So we have a bijection from Π to $\{0, 1, \ldots, n\}$ and therefore we have a bijection from $\mathsf{Tuples}(k, \Pi)$ to the set of k-values from $\{0, 1, \ldots, n\}$ that sum n (denoted $\mathsf{Tuples}(k, \{0, 1, \ldots, n\})$). The cardinality of $\mathsf{Tuples}(k, \Pi)$ is the same as the cardinality of $\mathsf{Tuples}(k, \{0, 1, \ldots, n\})$, and we can obtain the cardinality of $\mathsf{Tuples}(k, \{0, 1, \ldots, n\})$ as the binomial coefficient $\binom{n+k-1}{n}$ using the stars and bars method [5].

Further examples

We give some further examples of decomposition trees in this section.

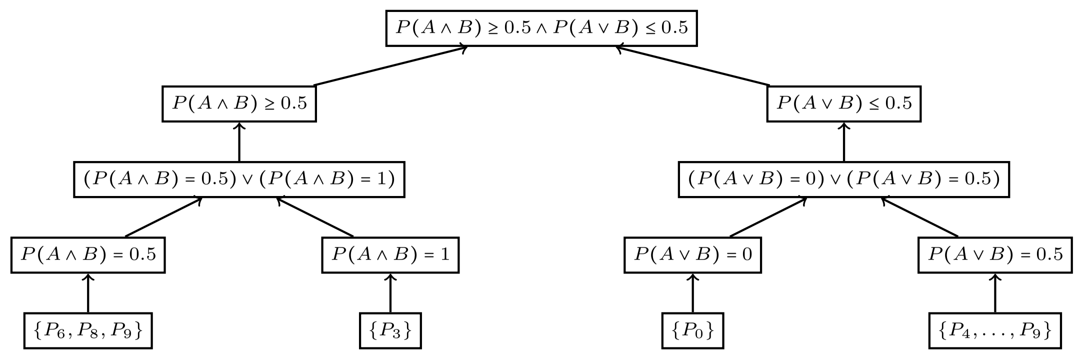

Fig. 3. A decomposition tree. Let P_1 to P_8 be defined as in Table 1. For the root, the set of models is $\{P_6, P_8, P_9\}$.

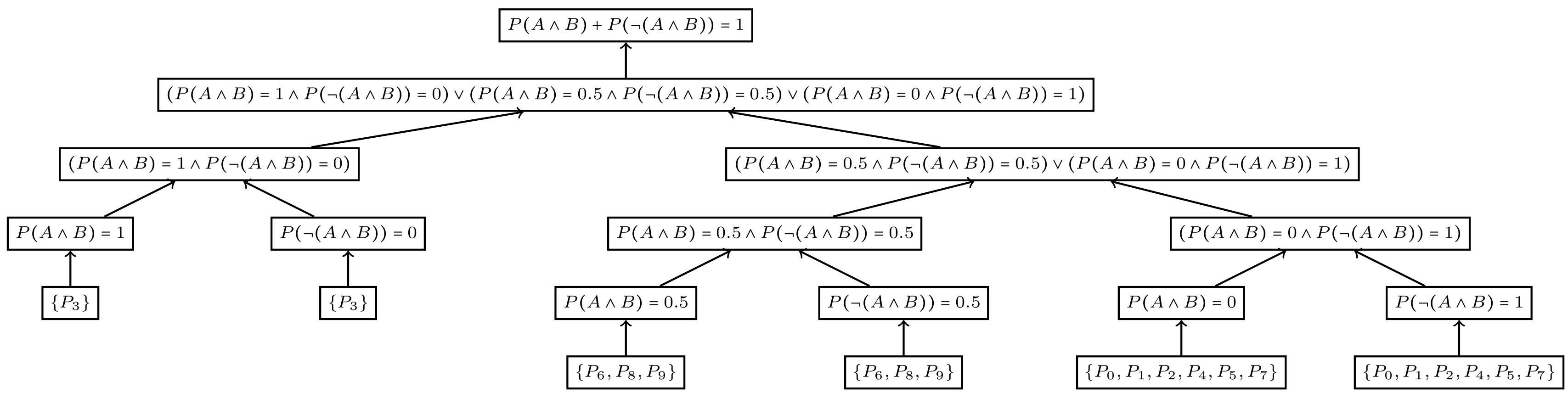

Fig. 4. A decomposition tree. Let P_0 to P_9 be defined as in Table 1. For the root, the set of models is $\{P_0, \ldots, P_9\}$.